MADHUBAN

BRAIN BUZZ

2

Mathemagical Tricks For Fun And Knowledge

Young Learners

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Mathemagical tricks for fun and knowledge

R K Murthi

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To my friend Biny Kurian

. . . For being one with me when it comes
to befriending numbers

Author's Note

Numbers hold lots and lots of fun; and quite a bit of magic too. But most of us never get a chance to play with numbers. While we are at school, we learn tables by rot; solve problems mechanically; memorize formulae. We groan and grunt, repeat Macaulay's statement, "Mathematics is as dry as dry biscuits."

We miss, once we get into this mindset, the fun and magic that go with numbers. For numbers are full of fun. They are good playmates.

What games can we play with them? How exciting are these games?

These two volumes, graded to suit the age groups 7 to 10 (Brain Buzz 1) and 10 and above (Brain Buzz 2), tell us just that. Therein lies the appeal of these books. They are fun to read. They take us along the fun-way to knowledge, an idea dear to me, and help us learn while we play.

Play with numbers. Befriend them. This friendship shall make life fun-filled for ever.

R K Murthi

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2002

1

SHORT CUTS TO CALCULATIONS

Sunita, Ranjan and I enter Grandpa's room and find him shuffling a pack of cards. We get closer and notice that they are not standard playing cards. They are special cards. They carry blocks of figures.

Grandpa hears our footsteps, lifts his eyes off the pack of cards and gives us a warm smile.

"Grandpa. These cards look so different," says Sunita after taking a look at them.

"They are magic cards," he has a smile on his lips, while we sit around him.

"Are they? If so we would like to have them too," I express my wish.

"But the cards would be of no use to you unless you know how to use them," says Grandpa.

"You will teach us the trick," Ranjan grins.

"I will. But before that, I want to tell you how to subtract quickly. The magic number that works here is 9," he gets ready to share the trick with us.

Magic 1: Quick Subtraction

"Subtract 458 from 1000," Grandpa says.

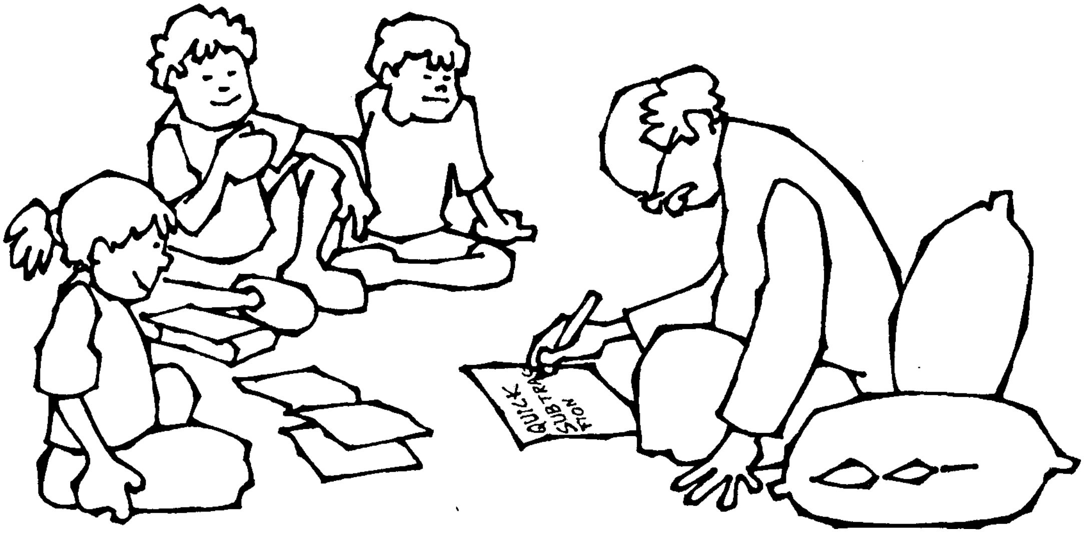
(We take 1000 as base because 458 is 3-digit number and hence the base must be 4-digit)

We do it by the conventional way. It works like this.

$$\begin{array}{r} 1{,}000 \\ -\ 458 \\ \hline 542 \\ \hline \end{array}$$

We get 542.

Grandpa explains a quicker way.



Here is the method.

The number to be subtracted is 458.

1. 4 is the digit at the hundreds place. Subtract it from 9. We get 5.

   $$9 - 4 = 5 \qquad \text{...(a)}$$

2. The digit in the tens place is 5. Subtract it from 9. We get 4.

   $$9 - 5 = 4 \qquad \text{...(b)}$$

3. Subtract the digit at the ones place from 10. We get 2.

   $$10 - 8 = 2 \qquad \text{...(c)}$$

   (Only the digit in ones place is subtracted from 10, rest from 9)

4. So 1000 – 458 = (a) (b) (c) = 542

Be a Magician

Subtract (i) 3,579 from 10,000 (ii) 53,621 from 1,00,000

Check your answers by using conventional method.

Magic 2: Vedic Method to Multiply

"How shall we multiply two numbers which are close to 100? Is there an easy way?" Grandpa chuckles to himself before telling us, "Let us find the product of 84 and 95."

He goes step by step.

84 = 100 –16	(84 is 16 less than 100)
95 = 100 – 05	(95 is 05 less than 100)

He reformats the above chart.

Primary	*Secondary*
84	– 16
95	– 05

Step 1. Find out the sum of the primary numbers,

i.e. 84 + 95 = 179

Step 2. Subtract 1 from the digit in the hundreds place.

$$\underset{-1}{1}79 \longrightarrow 79$$

We get 79 .... (A)

"But we get 79 by adding 84 and –05", says Sunita. "Or by summing up 95 and –16," says Ranjan.

"All roads lead to Rome," I joke.

"That's it. But don't forget to subtract 1 from the first digit of the sum when we work with just two primary numbers," grins Grandpa.

Step 3. Take the secondary numbers and find their product.

$$-16 \times -05 = 80$$

We get 80 .... (B)

"We are now ready with the answer. The product of 84 and 95 is (A) (B), i.e. 7980"

Note: If the product of secondary numbers is a single digit number, put a zero before the number. For example, consider the product of 97 and 98.

Sum of the primary numbers, 97 + 98 = 195

$$\begin{array}{l} 1\,95 \rightarrow 95 \\ -1 \end{array}$$

The product of secondary numbers, 3 × 2 = 6 (06)

∴ Product is 9506

Be a Magician

Find the product of the pair of numbers listed below.

(i) 89 and 97 (ii) 87 and 94

Magic 3: Base 1,000

"Find the product of 998 and 892," says Grandpa.

We do it by the usual method.

$$\begin{array}{r} 998 \\ \times\ 892 \\ \hline 1{,}996 \\ 89{,}82 \\ 7{,}98{,}4 \\ \hline 8{,}90{,}216 \\ \hline \end{array}$$

Grandpa shows us the Vedic way.

"These are numbers close to 1,000. So let me take the base as 1,000," he says. Then he writes down the following steps.

Primary	*Secondary*	
998	– 2	(998 is 2 less than 1000)
892	– 108	(892 is 108 less than 1000)

He adds 998 and 892, subtracts 1 from the digit in the thousands place. He gets 890.

Grandpa multiplies – 108 with – 2 and gets 216. He then says, "The product is 8,90,216," Grandpa smiles.

"Right," we react.

"Find the product of 118 and 104. Take base 100," says Grandpa and prepares the following chart.

Primary	*Secondary*
118	+ 18
104	+ 04

118 + 104 = 222

Reducing the digit at the hundreds place by 1 we get

$$\begin{array}{l} 2\,22 = 122 \\ -1 \end{array}$$

Now product of 18 and 4 is,

$$18 \times 4 = 72$$

So product is 12,272.

"Try product of 9,995 and 2,349, taking base 10,000," says Grandpa. He shows us how to proceed.

Primary	*Secondary*
9,995	– 5
2,349	– 7,651

By adding 9995 + 2349, we get 12344

Subtracting 1 from the digit at ten thousands place, we get 2344.

Multiplying –5 by – 7651, we get 38255.

So the number is 234438255

"But remember that the product cannot have more than eight digits," he says. Then he asks, "Why?"

We look at each other. We can't find the reason. So Grandpa explains, "Because 2,349 × 10000 = 2,34,90,000. So 2,349 × 9995 must be less than the above number. So write the results in such a way that the end product will have not more than 8 digits," Grandpa gets down to work.

"Write the number in the following way and then add," he says.

$$\begin{array}{r} 2344 \\ +\ 38255 \\ \hline 23478255 \\ \hline \end{array}$$

"This is the product. Check it by actual multiplication," says Grandpa.

Be a Magician

Use this method to find product of

(i) 1,788 × 9,998 (ii) 9,987 × 9,995

Magic 4 : Special Square

"For finding the square of two-digit numbers, ending with 5, there is an easy method," Grandpa leads us on. "Remember that the square of a number is obtained by multiplying the number by itself".

"How does the method work when we try to find the square of 45?" asks Sunita.

"We take the product of 4 and the next integer, i.e. 5. In this case, it is 4×5 which is 20. Now we take the square of the last digit, i.e. 5×5 which is 25."

"The square of 45, therefore, is 2025," says Grandpa

"Now, calculate the square of 85," Grandpa tells us.

The integer next to 8 is 9. So we take product of 8 and 9. We get 72. Now take the square of the last digit, i.e. 5. ~~It gives 25.~~ So the product is 7,225.

Magic 5 : Fraction to Recurring Decimal

"We know about recurring decimals which stand for fractions with denominators like 3 or 7. Vedic mathematics gives easy methods to convert fractions into recurring decimals. Recurring decimal is equivalent to a fraction where the denominator ends in 9 and the numerator is less than the denominator is the simplest," says Grandpa and then lays down the rules.

1. The length of the recurring portion will always be even.
2. The length will be one less than the denominator if it is a prime number ending in 9.

Examples:

Denominator	*Length of recurring portion of the decimal*
19	18
29	28
59	58

3. Break the recurring portion into two at the middle. Write the second bit under the first. Add. You will get a sequence of 9s.

Example:

Fraction $\frac{1}{19}$ = 0.052631578947368421 recurring. Note that the recurring portion had 18 digits. Split it in the middle.

Write the second section under the first.

$$\begin{array}{r} 052631578 \\ +\ 947368421 \\ \hline 999999999 \end{array}$$

"How does that help when it comes to finding the recurring decimal equivalent to a fraction?" we ask.

"Let me show you with the example above," Grandpa gets ready to convert the fraction $\frac{1}{19}$ into a recurring decimal.

He writes down the steps, while explaining.

Step 1. The numerator is less than the denominator. So the quotient will be less than 1. That tells us what to do. We place a 0 before the decimal point.

Step 2. Add 1 to the denominator. We get 20.

Step 3. Drop the 0. The divisor is 2.

$$2\overline{)1}^{\,0}$$

The numerator is 1. It is less than the divisor. Put a 0 next to it. At the same time write a 0 after the decimal. Now the problem looks like this.

$$\begin{array}{r} 0.05 \\ 2\overline{)\ 10} \\ \underline{10} \\ 0 \end{array}$$

After every division, write after the remainder the quotient that we got last. In this case the last quotient is 5.

$$\begin{array}{r} 2 \\ 2\overline{)\,05} \\ 4 \\ \hline 1 \end{array}$$

Decimal now becomes 0.052

Write the quotient 2 after the remainder. We get

$$\begin{array}{r} 6 \\ 2\overline{)\,12} \\ 12 \\ \hline 0 \end{array}$$

Decimal now becomes 0.0526

Write the quotient 6 after 0. We get 06 or 6. Divide by 2. Remainder is 0. Quotient is 3.

Decimal now reads 0.05263

"Divide 3 by 2. (Quotient is 1. Remainder is 1)

Decimal now reads 0.052831

Write quotient after remainder. We get 11. Divide 11 by 2.

The remainder is 1. The quotient is 5.

Decimal now reads 0.0526315.

Step 4. Continue till the sequence of figures start repeating itself. The sequence will, in this case, repeat after 18 digits. (The denominator is 19. The recurrence sequence will have a length of 19 – 1 i.e. 18).

Grandpa continues till he gets the recurring decimal

0.052631578947368421

Easier Method

"Can't we do this more easily?" we ask.

"There is a much simpler method. At each successive stage, get the quotients (Q) in this first lines; the remainders (R) in the next line.

Q : 0 5 (Divide 05 or 5 by 2. Quotient is 2, Remainder is 1)

Q : 0 5 2

R : 0 1

Divide 12 by 2. We get quotient 6. There is no remainder. The table reads :

Q :	0	5	2	6
R :	0	1	1	0

Continue till we get 1 on line Q and 0 to its left on line R. Then stop. The recurring portion starts at this point. We count the number of digits.

Q :	0	5	2	6	3	1	5	7	8	9	4	7	3	6	8	4	2	1
R :	0	1	0	0	1	1	1	1	0	1	0	1	1	0	0	0	0	

Read out the sequence in line Q. It gives the recurring decimal for

$\frac{1}{19}$ it is 0.052631578947368421

"How many digits are there?" Grandpa asks. We check. There are 18 digits in the recurring decimal. It is one less than the denominator 19, which ends in 9.

Still Easier Method

"But you don't have to go all that way, if you remember that the sequence can be split into two parts. Each part will be complementary to the other," Grandpa breaks the sequence in the middle.

Seciton 1. (First nine digits) 0.52631578

Section 2. (Last nine digits) 947368421

"Add the two," says he.

We get 999999999.

"So, once we get the first nine digits of the recurring portion of the

decimal, we can write the next nine digits by taking the complements of 9, taking the digits in sequential order," says Grandpa. In other words, we have to calculate only half the number of digits of the recurring portion of the decimal. The rest can be obtained by taking the complements of 9," says Grandpa.

Magic 6 : Where the Denominator Doesn't End in 9

"How will we find the recurring decimal equivalent to a fraction whose denominator doesn't end in 9?" we ask.

"Let us take $\frac{1}{13}$.

"We know the table for 13. 13 × 3 is 39," says Grandpa. "$\frac{1}{13}$ is same is $\frac{3}{39}$. Now follow the method we learnt when we convert $\frac{1}{19}$ into a recurring decimal. The divisor will be 4. The numerator will start with 30. The answer is 0.076923.

"The first three digits after decimal are 0.76.

"The next three digits are 923.

"Add ...

"We get 999.

"The rule of complement works," he says.

"But the length of the recurring portion, in this case, is 6. Should it not be 1 less than 13, i.e. 12?" Sunita asks.

"Ah, that rule is valid only when the denominator is a prime number ending in 9. In this case, we multiplied 13 by 3 to get 39. 39, therefore, is not a prime number. Hence this rule doesn't work," Grandpa explains.

Be a Magicium

Find recurring decimal equivalent to $\frac{1}{7}$ using the above method.

(Hint: Make the fraction into the equivalent where denominator ends in 9, i.e. $\frac{7}{49}$. Answer is 0.142857)

Magic 7 : Trachtenberg shows the Way

"Short cuts to calculations are always welcome. Vedic Mathematics provides us many short cuts. So does the Speed System of Mathematics, developed by Jakow Trachtenberg, of Russia. He worked as a mining engineer. For him, playing with numbers was fun. Let us try his method to find the square of a two-digit number, beginning with 5. Let us try with 56," says Grandpa.

He writes down the following steps.

The original number = 56

Step 1. Take the square of 5, i.e.	= 25	... (a)
Step 2. The last digit of the number	= 6	...(b)
Step 3. (a) + (b)	= 31	...(c)
Step 4. Square of the last digit	= 36	...(d)

The square of 56, therefore, is (c) followed by (d) i.e., 3,136.

Note: If the square of the last digit is a single digit number, put a zero before the number.

For example: Square of 53

Square of 5 = 25. Now 25 + 3 = 28

Square of the last digit = 9. So the square of 53 is 2809.

Magic 8 : Quick Roots

"There are quick ways to compute the square roots/cube roots of numbers which are perfect squares/perfect cubes. How does it work?" Grandpa moves on to the next trick. He demonstrates the procedure.

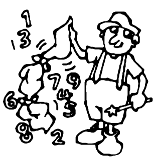
"Let us find the square root of 4761.

"Take the number and omit the last two digits. In this case it is 47. This lies between 36 and 49. So the square root must start with 6.

"The number ends in 1. So the square root must end with either 1 or 9.

"At this stage we can either calculate the square of 61 and 69 and confirm that 69 is the square root otherwise we can use common sense, that 4761 is closer to the square of 70 than to the square of 60 and correctly identify square root as 69.

"Let us now find the cube root of 19,683. Take the number. By omitting the last three digits, we get 19. We know that the cube root of 8 and 27 are 2 and 3 respectively. So the first digit of the cube root of 19,683 is 2. Take the last 3 digits. The last digit is 3. We know that the only number, whose third power ends in 3, is 7. So the cube root is 27.



Be a Magician

(i) Find the square root of 7056.

(ii) Find the cube root of 314,432.

Magic 9 : Ethiopian Show

"Do you know how Ethiopian tribes multiplied two numbers? They knew how to multiply and divide by 2. Nothing more," says Grandpa.

Then he shows us the method to calculate the product of 17 and 23. The steps he takes are as under.

	Column I	*Column II*
a.	17	23
b.	8	46
c.	4	92
d.	2	184
e.	1	368

In column I, the first number is halved till to get 1. In column II, the second number is doubled till to get the equal number of rows with column I.

He scores out every row where the number in Column I is even.

The chart now reads,

	Column I	*Column II*
a.	17	23
b.	1	368

Grandpa adds the numbers in Column II, i.e. 368 + 23.

$$368 + 23 = 391$$

He gets the total 391.

Grandpa says this is the product of 17 and 23. We check it by usual method and find it is true.

Be a Magician

Find the product of the following using the Ethiopian technique.

(i) 37 × 49 (ii) 28 × 34 (iii) 18 × 29

2

POWER PLAY

Magic 1 : Number 407

Grandpa asks us to write down the number 407. He says it is a unique number. We wonder why? He then explains.

$$\begin{array}{rcr}
4 \times 4 \times 4 & = & 64 \\
0 \times 0 \times 0 & = & 0 \\
7 \times 7 \times 7 & = & \underline{343} \\
\text{Total} & = & \underline{407}
\end{array}$$

Grandpa says, "The number is the sum of the cube of the digits of the number." Then he writes down the number 9,474 asks us to find out what is special about it.

"4 appears twice," Sunita jokes.

"That is no great discovery," I take a dig at her.

"Don't think you are Newton," she is quick to rap me.

Ranjan seems lost in calculations. We watch. He calculates the fourth power of the four digits of the number.

$$\begin{array}{rcr}
9 \times 9 \times 9 \times 9 & = & 6{,}561 \\
4 \times 4 \times 4 \times 4 & = & 256 \\
7 \times 7 \times 7 \times 7 & = & 2{,}401 \\
4 \times 4 \times 4 \times 4 & = & \underline{256} \\
\text{Total} & = & \underline{9{,}474}
\end{array}$$

"Ah! The number 9,474 is the sum of the fourth power of the digits of the number," all three of us share the fact with Grandpa.

Be a Magician

Test the following.

(i) 93,084 is the sum of the fifth power of its digits, i.e. of 9, 3, 0, 8 and 4.

(ii) 5,48,834 is the sum of the sixth power of its six digits.

(iii) 99,26,315 is the sum of the seventh power of its seven digits.

(iv) 8,85,93,477 is the sum of the eighth power of the eight digits in it.

(v) 91,29,85,153 is the sum of the ninth power of its nine digits.

(vi) 467,97,07,774 is the sum of the tenth power of its ten digits.

Magic 2 : Number 69

"69 is a unique number," says Grandpa.

"Why?" we ask.

"Its square is . . .," he works out the product on a sheet of paper. He gets 4,761. "Find out its cube," he tells us.

We do that. We multiply 4,761 by 69. We get 3,28,509.

"Examine the square and the cube," says Grandpa.

We do that. Then we notice that:

1. The square of 69 has four integers. The cube has six digits.
2. The square and the cube do not have any digit in common.
3. Between the two of them, the square and the cube of 69 use up all the digits from 0 to 9 once.

Magic 3 : Number 1001

"Look at 3,46,346. It is a unique number," says Grandpa. "So is 7,59,759. Can you find out its unique feature?"

We look at each other. "The first three digits of the numbers are the same as the last three digits," we tell him.

"That is obvious. But there is more to such numbers," says Grandpa.

He divides the number 3,46,346 by 7.

346346 ÷ 7 = 49478.

There is no remainder.

He divides 49,478 by 11. Still there is no remainder.

The quotient is 4498.

He divides 4,498 by 13.

Surprise of surprise, there is no remainder. The quotient is 346. "See. This number is divisible by 7, 11 and 13," he says.

"Look at the quotient 346. It is the first three digits of the original number," says he.

"It is also the last three digits of the number," we add.

"That's what makes the number unique," Grandpa grins.

"Try with 7,59,759. What do you get after successive divisions by 7, 11 and 13? The answer is 759. The quotient is the first three digits of the original number and also the last three digits of the original number.

"There is a magic in every 6-digit number in which the first three digits repeat in the same order and form the last three digits.", he says.

"But why is it so?" we ask.

"That is where the magic lies," Grandpa peers at us.

"We see no magic," we groan.

"I will tell you where the magic lies. Multiply 7 by 11. We get 77. Multiply 77 by 13. What do we get? It is 1,001," says Grandpa.

We nod our heads. But the magic is still not evident.

"Look at the number 346,346. It is actually 346,000 + 346, i.e.

$$\begin{array}{r} 346000 \\ +\ 346 \\ \hline 346346 \end{array}$$

In other words, $(346 \times 1000) + (346 \times 1)$

$$= 346000 + 346 = 346346$$

$$\text{Or } 346 \times (1001) = 346346$$

"Hence any 6-digit number, whose first three digits repeat at the end, is always divisible by 1,001 or its factors 7, 11 and 13. When we divide it successively, by the three numbers, 7, 11 and 13, (in any order) we get the 3-digit number which repeats itself in the 6-digit number."

Be a Magician

Divide the following numbers by 7, 11 and 13 successively.

(i) 3,42,342 (ii) 9,87,987 (iii) 2,10,210 (iv) 6,39,639

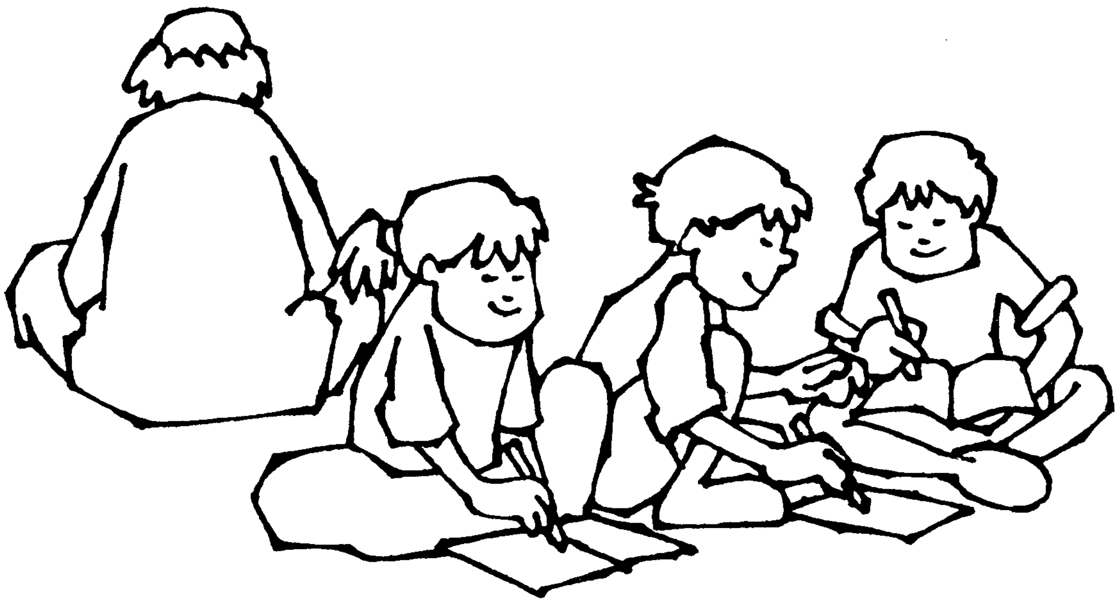
Magic 4 : 2001's Trick

"Want to play a new trick? Come on, then," says Grandpa.

He then asks us to write down on a sheet of paper any 2-digit number.

"Don't show it to me. Yet I will give you the number if you perform these multiplications," he says.

We agree.



"Multiply the number by 3. And, don't give me the product. Not now," Grandpa warns us.

We get the product.

"Multiply the product by 23," says he.

That takes some time. But we put our heads together. Has it not been said, "Two heads are better than one"? Here we are, putting three heads together. So we find the product. We turn to Grandpa. He puts his finger on his lips and says, "No. I don't want this product either. But, please, multiply the final product by 29", he looks pleased with himself.

We get down to the job. Finally we get the product.

"Give me the number," says he.

We read out the number 64,032.

"Ah, you started with the number 32," says Grandpa instantly.

"How did you find out?" we ask.

"Quite simple. It is the number formed by the last two digits," he says.

"Is it always so?" we ask.

"Yes."

"Why?"

"For that you must once again do some multiplication. Multiply 23 by 3. What do you get? 69. Right?" he asks.

"Yes."

"Now take the product of 69 and 29," he makes us find the result.

$$69 \times 29 = 2001$$

"So when you picked up 32 and multiplied it by 3, 23 and 29, you were actually multiplying the original number by 2,001.

$$2{,}001 = 2{,}000 + 1$$

"So the last two digits of the final product that you get after multiplying your chosen number by 3, 23 and 29 will be the chosen number itself," Grandpa finds us bouncing with joy.

Magic 5 : Numbers 1,233 and 8,833

"1,233 is sum of squares of 12 and 33," says Grandpa, writing down the calculations as below.

12 × 12	=	144
33 × 33	=	1,089
Total	=	1,233

"Try this with 8,833," Grandpa leans back while we get down to work.

88 × 88	=	7,744
33 × 33	=	1,089
Total	=	8,833

"Wonderful," we jump with joy.

"There is more magic that the numbers 1,233 and 8,833 hold. Multiply 1,233 by 8,833. What is the product?" Grandpa makes us find the product.

We do that and get the product. It is 1,08,91,089.

"Anything unique about it?" Grandpa asks.

We examine the number. We can't believe it. 1089 is the first half of the number. It is also the last half of the number. "1,089 has magic in it," we turn to Grandpa.

"It is the square of 33," he points out.

Magic 6 : Number 65

Grandpa prepares the following chart while we watch.

$$4 \times 4 + 7 \times 7 = 65$$
$$1 \times 1 + 8 \times 8 = 65$$

"Know what? 65 is the smallest number which can be expressed as the sum of squares of two integers in two different ways," he tells us.

Magic 7: Number 1105

$$4 \times 4 + 33 \times 33 = 16 + 1089 = 1105$$
$$9 \times 9 + 32 \times 32 = 81 + 1024 = 1105$$
$$12 \times 12 + 31 \times 31 = 144 + 961 = 1105$$
$$23 \times 23 + 24 \times 24 = 529 + 576 = 1105$$

Be a Magician

Find the smallest number which can be expressed as the sum of two squares in eight/sixteen different ways.

(This was posed through Mindsport, The Times of India)

The answer was logically arrived by Avinash Saralkar through the following steps.

$$65 = 1 \times 1 + 8 \times 8$$

$$65 = 4 \times 4 + 7 \times 7$$

$$65 = 5 \times 13$$

[Note that $5 = 1 \times 1 + 2 \times 2$

$13 = 2 \times 2 + 3 \times 3$]

Similarly, $1105 = 5 \times 13 \times 17$

Note that $17 = 1 \times 1 + 4 \times 4$

Also note that 5, 13, and 17 are all odd numbers.

"Which is the next odd number that can be represented as the sum of two squares in 8 different ways." Grandpa asks.

The perfect squares before 23 are 1, 4, 9 and 16. No two numbers add up to 23. So go to the next odd number. It is 29. It is 25 + 4,

i.e. $5 \times 5 + 4 \times 4$.

That leads us to $32045 = 5 \times 13 \times 17 \times 29$.

"Take sum of $5 \times 5 + 13 \times 13 + 17 \times 17 + 29 \times 29$" he says.

We get 32, 045 and become happy.

"Test and confirm that this number can be expressed as the sum of two squares in 8 different ways" Grandpa gives us the task.

Note: Remember that the factors of each of the numbers with this unique feature are themselves sum of two squares.

Magic 8 : Number 1,729

"It is the smallest number which can be expressed as the sum of two cubes of whole numbers in two different ways," says Grandpa and writes down the following chart.

$1729 = (12 \times 12 \times 12) + (1 \times 1 \times 1)$

$1729 = (10 \times 10 \times 10) + (9 \times 9 \times 9)$

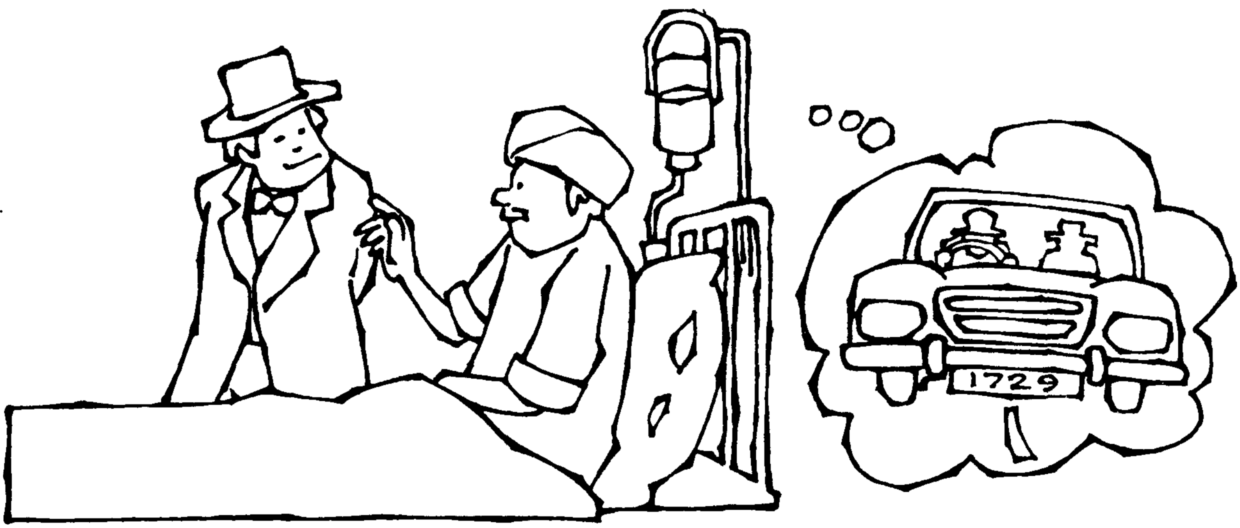
"There is a very interesting story behind this," says Grandpa.

"Ramanujan, the eminent Indian mathematician went to Britain. There he did research in various fields of Mathematics. His friend and guide was Professor Hardy.

"Once Ramanujan was taken ill. He had to spend a few days in a local hospital. Professor Hardy came to the hospital to enquire about his health.

"Ramanujan greeted Professor Hardy. After Professor Hardy sat down, the two started talking.

"Professor Hardy told him, during the conversation: 'I came by cab. Its number was 1729.'



"Ramanujan's eyes opened wide. He shot out: 'Do you realise that it is the smallest number that can be expressed as the sum of two cubes in two different ways.'

"That shows how alert he was to the magic of numbers!" Grandpa points out.

Magic 9 : Ramanujan's Show

Grandpa took note of the fact that 3 is square root of 9.

He wrote it as

3 = square root (SR) of (1 + 8)

= SR of (1 + 2 × 4) ...(A)

= SR of (1 + 2 × SR of 16)

= SR of {1 + 2 × SR of (1+15)}

= SR of {1 + 2 × SR of (1 + 3 × 5)} ...(B)

= SR of {1 + 2 × SR of (1 + 3 × SR of 25)}

= SR of [1 + 2 × SR of {1 + 3 × SR of (1 + 24)}]

= SR of [1 + 2 × SR of {1 + 3 × SR of (1 + 4 × 6)}] ...(C)

"Do you find anything unusual here?" Grandpa asks.

We examine the chart. Then we see light. The sequence can be extended to any length.

At stage A, we get at the end square root of 1+ 2 × 4.

At stage B, it is square root of 1 + 3 × 5.

At stage C, it is square root of 1 + 4 × 6.

So we can expect at stage D the end form as square root of 1 + 5 × 7 and at stage E the end form as square root of 1 + 6 × 8, etc.

"Good," Grandpa pats us.

Magic 10 : More about Ramanujan's Show

"Here is more about Ramanujan," Grandpa sets out once again. He noticed that

$$\frac{1}{4} + 2 = \left(1 + \frac{1}{2}\right) \times \left(1 + \frac{1}{2}\right)$$

$$\frac{1}{4} + 2 \times 3 = \left(1 \times 2 + \frac{1}{2}\right)\left(1 \times 2 + \frac{1}{2}\right)$$

$$\frac{1}{4} + 2 \times 3 \times 5 = \left(1 \times 5 + \frac{1}{2}\right) \times \left(1 \times 5 + \frac{1}{2}\right)$$

$$\frac{1}{4} + 2 \times 3 \times 5 \times 7 = \left(2 \times 7 + \frac{1}{2}\right) \times \left(2 \times 7 + \frac{1}{2}\right)$$

"How is that?" Grandpa asks.

"Excellent!" we cheered.

Magic 11 : Number 49

"Look at 49. It is 7 × 7. So it is a perfect square," says Grandpa.

We nod our heads.

"Look at the digits of 49. They are 4 and 9. 4 is 2 × 2. So it is a perfect square. So is 9, which is 3 × 3. This uniqueness is shown by many numbers," Grandpa jots down a few numbers.

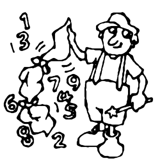
169 : 16, 9 and 169 are squares of 4, 3 and 13.

3,249 : 324, 9 and 3249 are squares of 18, 3 and 57.

93,025 : 9, 3025 and 93025 are squares of 3, 55 and 305.

These unique numbers, (each number is a square and their split parts are also squares) always end with 9 or 25.

(Science Today, April 1982, reported by an eminent mathematician D R Kaprekar)



Be a Magician

Test the numbers 64009 and 81225.

Magic 12 : Numbers 144 and 441

"144 is the square of 12. 441 is the square of 21. Can you spot something unique in it? 12 and 21 share the same digits in reverse orders. So do their squares 144 and 441. So is the case with 13 and 31. Their squares are 169 and 961," says Grandpa.

Be a Magician

Find the squares of the following pairs of numbers.

113, 311; 103, 301; 112, 211; 102, 201; 1122, 2211; 1022, 2201.

See how the same digits in different orders occur in their squares too.

Such numbers are countless. Can you form more such pairs?

Magic 13 : Numbers 11,826 and 30,384

Grandpa writes down the numbers 11,826 and 30,384.

"They are unique," says he. "Square the first number. The square is 13,98,54,276. It contains the digits 1 to 9, just once each. Square the second number. We get 92,31,87,456. Here too, the numbers 1 to 9 occur, once each," says Grandpa.

Magic 14 : Numbers 24, 32, 40, etc.

Grandpa shows us the series of numbers. Then he says, "Add up the numbers in the series, up to any point, and add 1. You get the square of an odd number. Here is how it looks."

Sum of the first two numbers 8 + 16 = 24

$$24 + 1 = 25$$
$$25 = (5)^2$$

Sum of the first three numbers 8 + 16 + 24 = 48

$$48 + 1 = 49$$
$$49 = (7)^2$$

"Try as far as you want. The law remains valid," says Grandpa.

Magic 15 : Numbers 0, 1, 512, etc.

"What's unique about the numbers 0,1, 512, 4913, 5832, 17576 and 19683?" Grandpa asks.

We fail to spot the magic.

"Their cube roots are the sum of the digits of the numbers," says Grandpa.

We test with 512.

$$5 + 1 + 2 = 8$$
$$8 \times 8 \times 8 = 512$$

"That's true Grandpa, says Sunita. "Let's take the number 4913", says Grandpa.

$$4 + 9 + 1 + 3 = 17$$
$$17 \times 17 \times 17 = 4913$$

Magic 16 : Strange Squares and Cubes

"Watch. I shall work on some numbers. They show magic when we take their squares or cubes," says Grandpa. We watch while he prepares the chart below.

The square of 45, 55, 703 and 2,223 given as follows.

$$45\ (20 + 25) = 2{,}025$$

$$55\ (30 + 25) = 3{,}025$$

703 (494 + 209) = 4,94,209

2,223 (494 + 1729) = 49,41,729

"Can you see the magic? The square of every number contains the numbers in the brackets which together make up the number. The numbers occur in the same order."

The square of 55, i.e.

$$55 \times 55 = 3025$$

$$55 = 30 + 25$$

Then Grandpa prepares the chart for cubes.

The cube of 18, 26, 27 and 45 given as follows.

18 (5 + 8 + 3 + 2) = 5,832

26 (1 + 7 + 5 + 7 + 6) = 17,576

27 (1 + 9 + 6 + 8 + 3) = 19,683

45 (9 + 11 + 25) = 91,125

"The cube of 26, made up of 1, 7, 5, 7 and 6, is 17,576," Grandpa finds us terribly excited.

"That's great," we scream.

"I refer to it as power play," Grandpa jokes.

Magic 17 : Power Play

"You know that any number multiplied by itself gives us its square. When we multiply a number by itself twice, we get its cube."

"Now look at this chart," he starts writing.

$2 \times 2 \times 2$ is 8. 8 is the cube of 2.

So 8 is 2 raised to the power 3 or 2^3.

16 is 2 raised to the power 4 or 4^2.

243 is 3 raised to the power 5 or 3^5.

Let us divide 243 by 27. We know that

$$243 = 3 \times 3 \times 3 \times 3 \times 3$$

$$27 = 3 \times 3 \times 3$$

$$\frac{243}{27} = \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3 \times 3}$$

$$= 3 \times 3 = 9$$

= 3 raised to the power 2 or 3^2.

"Let us look at it afresh."

243 = 3 raised to the power 5 [3^5]

27 = 3 raised to the power 3 [3^3]

9 = 3 raised to the power 2 [3^2]

"So 3 raised to the power 5 divided by 3 raised to the power 3 is 3 raised to the power 2, i.e. $\frac{3^5}{3^3} = 3^2$. This power is the difference of the powers 5 and 3. That is where power play is seen at its best," Grandpa is all smiles.

Be a Magician

Fill the blanks.

1. 7 raised to the power 10 divided by 7 raised to the power 8 = 7 raised to the power .....2.......
2. 5 raised to the power 9 divided by 5 raised to the power 7 = 5 raised to the power ....2...

3. 3 raised to the power 7 multiplied by 3 raised to the power 2 = 3 raised to the power ....

Magic 18 : Zero Power

"Zero has immense power. It is a great leveller," says Grandpa.

"Why do you say so?" we ask.

"What do we get when 2 raised to the power 3 is divided by 2 raised to the power 3? We can find the quotient in two different ways." He then writes down the two methods.

Method 1. Apply the formula given in magic 17.

$$\frac{2^3}{2^3} = 2^{3-3} = 2^0$$

So we get the quotient as 2 raised to the power 0.

Method 2. We also know that 2 raised to the power 3 is 8.

The number to be divided is 8. So is the divisor. So the quotient, in the case of 2 raised to 3 divided by 2 raised to 3 is $\frac{8}{8}$ or 1.

"Can you make something out of it?" Grandpa asks.

"You mean 2 raised to the power 0 is 1?" we can't believe that.

"It is true. It works always. Try it with 5. Take the fifth power of 5. Divide it by its fifth power. The formula gives us the result 5 raised to the power 0. Actual calculation gives us 1. The two must be equal. So 5 raised to the power 0 is also 1. Any number raised to the power 0 is 1. So when a number gets power 0, it is levelled down to 1," says Grandpa.

We see the truth in what he says.

"I can prove how great a leveller 0 is by another example," Grandpa waits for our reaction.

"So 0 is an imp," says Sunita.

"It is. Let me prove that. Start with number 'a' which is greater than 'b'. Then he starts preparing the chart that runs as under.

$$a = b + c$$

Multiply both sides by (a – b)

$$a(a - b) = (a - b) \times (b + c)$$

$$a \times a - ab = ab + ac - b \times b - bc$$

$$a \times a - ab - ac = ab - b \times b - bc$$

$$a \times (a - b - c) = b(a - b - c)$$

Divide both sides by (a – b – c)

We get $a = b$

"This is absurd. For we stated at the very beginning that 'a' is greater than 'b'.

"But you proved just now that a = b," we are taken aback.

"Know the reason. For that we have to look at the divisor we used", says Grandpa.

"It is a – b – c. But since a = b + c, a – b – c is 0. In other words, we have divided by 0.

"But we know that division by 0 is not possible," I point out.

"You said it. That's why we got an absurd result a = b."

"Here is another trick that shows how 0 levels down numbers. Multiply any number by 0. And we get 0," says Sunita.

"0 is the ace in the pack," says Ranjan.

3

SQUARE CARD AND RATIOS

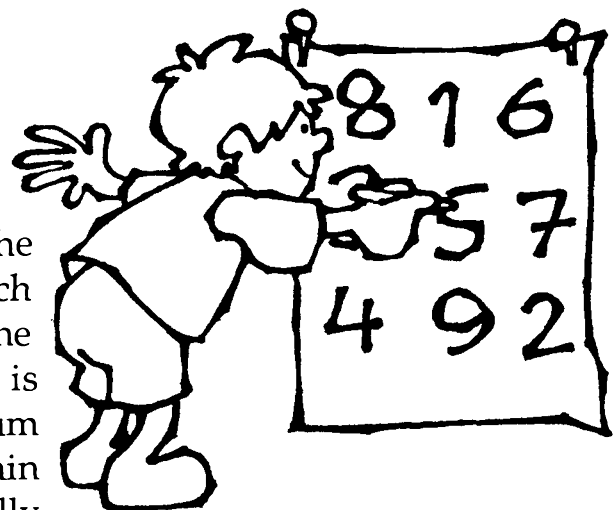
Magic 1 : 3 × 3 Magic Square

"What's the next trick?" Sunita pops up with another question.

"Shall I introduce you to magic squares," asks Grandpa.

Then he prepares a 3 × 3 square and fills it with numbers 1 to 9 at random. Or so it seems to us.

8	1	6
3	5	7
4	9	2



He then tells us, "Look at the 3 × 3 square given above. Each block in the square has one of the digits from 1 to 9. No digit is repeated. Add each row. The sum is 15. Add each column. Again the sum is 15. Add diagonally. The sum, once again, is 15. "Ah! That is magic indeed", We cry aloud.

"Such squares are called magic squares", Grandpa continues.

"These first appeared in China, in the 4th Century. The Chinese call them Loshu."

"Can we prepare a magic square Grandpa?" asks Sunita.

"Yes, of course, you can. It is quite simple," says Grandpa.

"But with which number must we start to build a 3×3 magic square?" Ranjan asks.

"That's a good question. Suppose you have to build a 3×3 magic square where the sum is 45."

Here are the following steps.

Take the desired sum. [45]

Step 1. Subtract 15 from the desired sum, i.e. 45.

$$[45 - 15] = 30 \qquad \text{...(A)}$$

[Because 15 is the sum of the first magic square we defined.]

Step 2. Divide A by 3 $[30 \div 3 = 10]$ ...(B)

Step 3. Add 1 to B $[10 + 1 = 11]$

Thus, the magic square that gives the total 45 starts with 11.

We have to follow these steps everytime we want to know the starting number of 3×3 magic square.

"Now let us prepare the magic square the sum of whose rows or columns or diagonals gives us 33," says Grandpa.

"We can calculate the starting number," says Ranjan.

Step 1. $33 - 15 = 18$

Step 2. $18 \div 3 = 6$

Step 3. $6 + 1 = 7$

So the magic square starts with number 7.

"Now you have to write this number in 3×3 block," says Grandpa.

		Column 1	*Column 2*	*Column 3*
		A	*B*	*C*
Row 1	P			
Row 2	Q			
Row 3	R			

"But where should we write 7 in the block?" asks Sunita.

"For this, you have to follow some rules. Let me set down the rules first," says Grandpa. Then he starts explaning the rules.

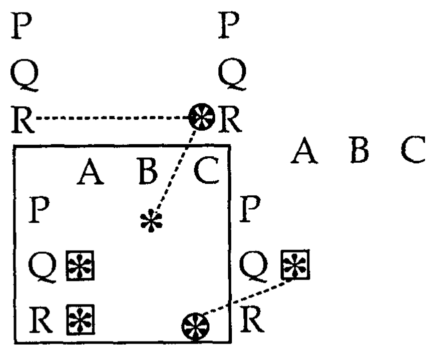
1. First of all treat the rows and columns as occurring cyclically.

2. Always start with the block at the centre of the top row, i.e. **Row 1 or Column 2.** Row 1 or Column 2 is PB.

3. Always move diagonally upwards.

   Suppose we are writing number in block PB. Moving diagonally upwards takes to RC.

   But we will write the number in the same block, i.e. 3 × 3 square.



   Now we have written the number in RC. Move diagonally upwards, it is QA. Write the number in QA is 3 × 3 square. And so on.

4. When we are moving diagonally upwards, sometimes the block is already occupied. Then drop down by one block.

For example we have written the number in QA. Diagonally upwards lands up to PB, which is already occupied. So drop down one block, i.e. from QA to RA and write the next number in RA.

These are the rules for 3 × 3 magic square. You must follow them strictly", Grandpa says.

"Now let's prepare 3 × 3 square where starting number is 7", we all shouted.

1. We start with block PB. Write 7 here.
2. Moving diagonally upwards leads us to RC. So write number 8 in the block RC.
3. QA is diagonally above RC. So write number 9 in the block QA.
4. Moving diagonally upwards from QA, we get PB which is already occupied. So drop down by one block. Write 10 in block RA.
5. Then write 11 in block QB, 12 in PC.
6. PC leads to RA which is already occupied. So drop down by one block. Write 13 in block QC.
7. In the same way we write 14 in block PA and 15 in RB.

"The square is now complete," says Grandpa.

The general table for the 3 × 3 square is given as follows.

a + 7	a	a + 5
a + 2	a + 4	a + 6
a + 3	a + 8	a+ 1

a = Starting number

Be a Magician

1. Which is the starting digit of the 3 × 3 magic square whose rows, columns or diagonals add to

   (i) 30 (ii) 75 (iii) 24 (iv) 48

2. Prepare on 3 × 3 magic square whose rows, columns or diagonals add to

(i) 45 (ii) 36 (iii) 72

Magic 2 : Ramanujan's Magic Square

"Ramanujan found the following magic square," says Grandpa. He then prepares the following magic square.

22	12	19	93	?	22	12	18	87	
28	100	3	15		88	17	9	25	
66	20	23	37		10	24	89	16	
30	14	101	1	s=146	19	86	23	11	s=139

"Note that all numbers are distinct. But they are not in sequence, as in the usual magic squares. The total, whether you add the numbers in a row or a column or a diagonal, is 146.

"Notice that in each of rows 1, 2 and 4, we find two numbers which add to 115 and the other two numbers to 31. The fourth line is adjustable. So you can get any total you like.

"For example, if we substitute the third row by

56 10 13 27

we get a magic square whose total is 136," says Grandpa.

Be a Magician

Test whether you can start with any four distinct numbers on the top row, form two pairs of numbers and then prepare a magic square with distinct numbers that gives you the desired total.

Magic 3 : Magic Square Does Magic

"I can play a trick with magic square", says Grandpa.

"Trick in a magic square! But how?" asks Sunita.

"Choose any number in a magic square. Score out the row and the column that contains it," instructs Grandpa.

Ranjan picked up the number 16.

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

"Now pick another number that has not been scored out. Go on till you pick 4 numbers." says Grandpa.

We take some time to work out the problem.

"The total of four number is 34," says Grandpa.

"How do you know that" we all shouted.

"Because I know that rows, columns or diagonals add up to 34 in this magic square.

"So the total of four numbers will always be the total of magic square," says Grandpa. "We will try this trick with our friends", we said.

Magic 4 : Pack of Show

That reminds me of the strange pack of cards whose magic Grandpa has promised to show us.

"Can we return to the pack of cards?" I ask.

"Why not? These cards will help you perform a trick. You can find the 2-digit number that someone has in mind, but is not known to you," Grandpa picks up the cards and continues, "here are seven cards. Each one of them contains a table of numbers."

Then he hands the cards over to us. Each card has a jumble of figures on it. The logic behind the distribution of the numbers on the cards is not readily seen.

$2^0 =$

Card 1

1	3	5	7	9	11	13	15	17	19
21	23	25	27	29	31	33	35	37	39
41	43	45	47	49	51	53	55	57	59
61	63	65	67	69	71	73	75	77	79
81	83	85	87	89	91	93	95	97	99

$2^1 = 2$

Card 2

2	3	6	7	10	11	14	15	18	19
22	23	26	27	30	31	34	35	38	39
42	43	46	47	50	51	54	55	58	59
62	63	66	67	70	71	74	75	78	79
82	83	86	87	90	91	94	95	98	99

$2^2 = 4$

Card 3

4	5	6	7	12	13	14	15	20	21
22	23	28	29	30	31	36	37	38	39
44	45	46	47	52	53	54	55	60	61
62	63	68	69	70	71	76	77	78	79
84	85	86	87	92	93	94	95	–	–

$2^3 = 8$

Card 4

8	9	10	11	12	13	14	15
24	25	26	27	28	29	30	31
40	41	42	43	44	45	46	47
56	57	58	59	60	61	62	63
72	73	74	75	76	77	78	79
88	89	90	91	92	93	94	95

$2^4 = 16$

Card 5

16	17	18	19	20	21	22	23	24	25
26	27	28	29	30	31	48	49	50	51
52	53	54	55	56	57	58	59	60	61
62	63	80	81	82	83	84	85	86	87
88	89	90	91	92	93	94	95	–	–

$2^5 = 32$

Card 6

32	33	34	35	36	37	38	39	40	41
41	42	43	44	45	46	47	48	49	50
52	53	54	55	56	57	58	59	60	61
62	63	96	97	98	99	–	–	–	–

$2^6 = 64$

Card 7

64	65	66	67	68	69	70	71	72	73
74	75	76	77	78	79	80	81	82	83
84	85	86	87	88	89	90	91	92	93
94	95	96	97	98	99	–	–	–	–

"Think of a 2-digit number. Don't tell me. Just pick up the cards on which your number appears," Grandpa gives us a task.

We do that. We hand over to him cards 1, 2, 3, 4 and 6. He takes just a second before telling us, "Your number is 47."

We can't believe our ears. He has got it right. But how?

"Well, the number is the sum of the first numbers on each of these five cards. What are the numbers? They are 1, 2, 4, 8 and 32. They add up to 47," Grandpa's wrinkles become more sharp.

We try the trick. Sunita thinks of a number, picks out the cards that contain the number. And Ranjan and I total up the first numbers on the cards and give her the right answer.

"Ah, I now know. The magic lies in the cards," I shout in excitement.

"That, we know. But how do we prepare the cards. That's where numbers come into play. Let us analyse each card," Grandpa lists out the following clusters of continuous numbers.

Card 1: (1), (3), (5), (7) ....

Card 2: (2,3), (6,7), (10, 11) ....

Card 3: (4,5,6,7), (12,13,14,15), ...

Card 4: (8 to 15), (24 to 31), (40 to 47) ....

Card 5: (16 to 31), (48 to 63), (80 to 95)

Card 6: (32 to 63), (96 to 99)

Card 7: (64 to 99)

"Can you see a pattern? The number of numbers in a cluster on a card is decided by the first number of the card. For example in **Card 1**, the first number is 1 so there is only one number in a cluster (1) (3) (5) (7). **In Card 2**, the first number is 2 so there are only two numbers in the cluster (2, 3) (6, 7), (10, 11). The difference between the last number of a cluster and the first number of the next cluster is 1 more than the first number of the càrd."

In Card 2, First cluster is (2, 3). Next cluster is (6, 7)

Last number of (2, 3) is 3. First number of the cluster (6, 7) is 6.

Difference = 6 – 3 = 3

3 is 1 more than 1st number of (2, 3)

In Card 3

First cluster is (4, 5, 6, 7). Next cluster is (12, 13, 14, 15)

Last number of the cluster (4, 5, 6, 7) is 7.

First number of the cluster (12, 13, 14, 15) is 12.

Difference 12 – 7 = 5

5 is 1 more than 1st number of (4, 5, 6, 7)

In **Card 3** the first number is 4. So 4 numbers appear in sequence in the first cluster. The numbers are 4, 5, 6, 7.

"What is the first number of the next cluster? It is 12", says Grandpa.

"But how?" we all shouted together

"Let me explain," Grandpa now writes down the steps.

First number of the first cluster = 4

Last number of the first cluster = 7

So first number of second cluster = 7 + 4 + 1 = 12

"But why do you add only the first numbers of the selected cards?" we turn to Grandpa.

"Every number can be expressed as the sum of some of the numbers of the sequence 1, 2, 4, 8, 16, etc. Each number in the sequence is double the previous number. These are the numbers that form the first digits of the seven cards," explain Grandpa.

Carry out several tests, with one or other of you thinking up a number and picking up the right cards, and the others giving him/her the right answer.

"See the magic of mind reading," says Grandpa.

"But, in this case, we know that it is logic, not magic that helps you in mind reading," we tell him.

"But others think you can read their minds," Grandpa chuckles.

"Any another trick that makes it look like mind reading?" we ask.

Magic 5 : Fibonacci Magic

"Here is another magic that I want to share with you," Grandpa stands with his back to us and says, "Write down a number. Any number will do. But for practice, let us start with a single digit

number."

Ranjan writes 7.

Grandpa asks me to write another number under it.

I write 5.

"Add up the two numbers. Let it be the third number in the chart." Sunita does that.

The chart, at this stage, reads as under.

$$\begin{array}{r} 7 \\ +\ 5 \\ \hline 12 \\ \hline \end{array}$$

"Now add the last two numbers," says Grandpa. "Write that below the last number. Do this again and again till you have ten numbers in the chart," he say.

We work at the chart.

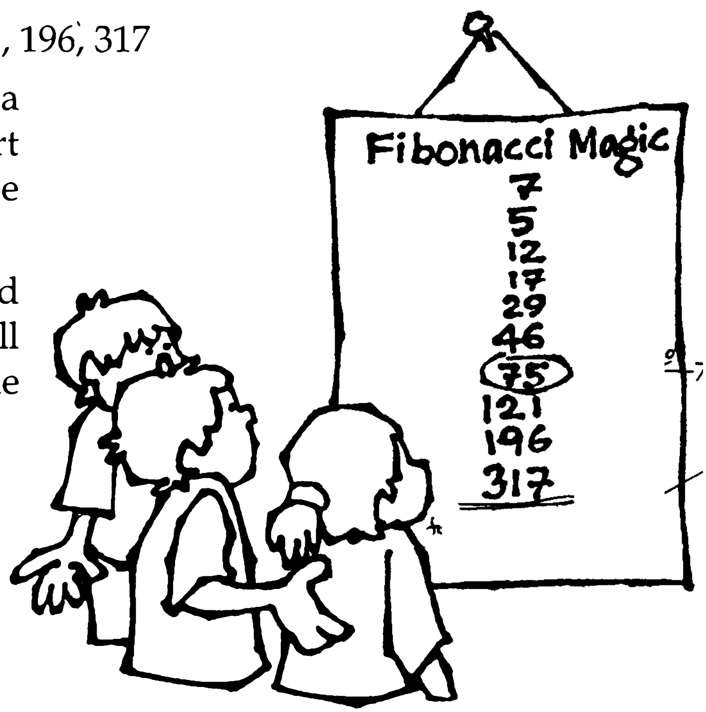
It finally reads like this :

7, 5, 12, 17, 29, 46, 75, 121, 196, 317

Grandpa turns, takes just a second to look at the chart and says, "The total will be 825."

Is he bluffing? No. We add the numbers. It takes us full five minutes. We get the total as 825.

"How did you guess? You couldn't have added up those digits that quickly," I point out.



"I just looked at the 7th number in the chart. And, lo presto! I k the total," Grandpa speaks in circles.

"Why the seventh number?" Sunita asks.

"Because it holds the key. You know that the first two numbers are chosen at random. I didn't even know what they were. Each subsequent number in the chart is the sum of the two previous numbers. In such a case, the total of the first ten digits is always 11 times the number at the 7th place," he says.

"But why?" Ranjan is the first to raise the question.

"Ah, let me explain," Grandpa sits with us and prepares the chart, taking 'a' and 'b' as the starting numbers. The chart reads.

First number	a	
Second number	b	
Third number	a + b	($1^{st} + 2^{nd}$)
Fourth number	a + 2b	($2^{nd} + 3^{rd}$)
Fifth number	2a + 3b	($3^{rd} + 4^{th}$)
Sixth number	3a + 5b	($4^{th} + 5^{th}$)
Seventh number	5a + 8b	($5^{th} + 6^{th}$)
Eighth number	8a + 13b	($6^{th} + 7^{th}$)
Ninth number	13a + 21b	($7^{th} + 8^{th}$)
Tenth number	21a + 34 b	($8^{th} + 9^{th}$)

"Add the numbers," says Grandpa.

We get 55a + 88b

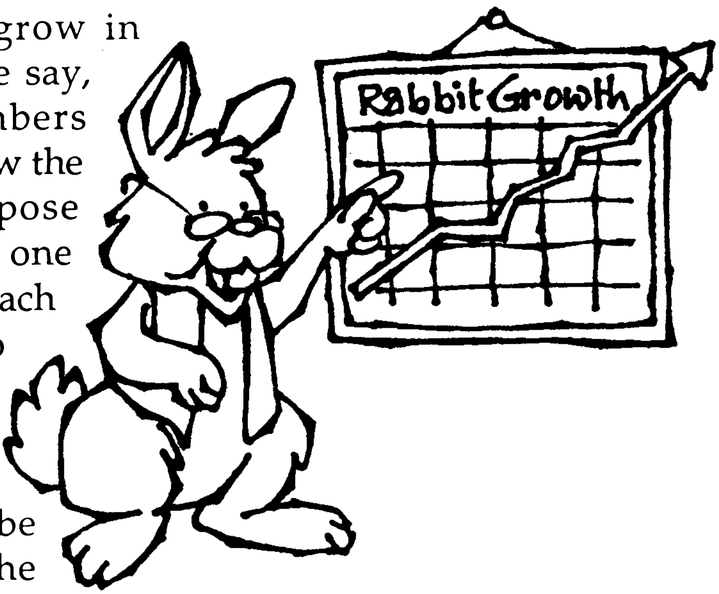
"The seventh term is 5a + 8b," Grandpa points out.

Now we see the secret. The total of the first ten terms of such a series is always 11 times the seventh number. So one has to only note the seventh term and multiply it by 11 to get the total.

"The above magic is the offshoot of a series noticed first by Leonardo Fibonacci. His series is 1, 2, 3, 5, 8, 13, 21, etc. Each number in the series, from the third term onwards, is the sum of the two previous

numbers. Botanists studied arrangements of leaves on the stems of plants and trees; counted the whirls on pine cones; the petals in sunflowers; the spirals of snail shells. All these are one of the numbers of the Fibonacci series. The botanists refer to this as PHYLLOTAXIS," says Grandpa.

Magic 6 : Population Growth



"Rabbits, it is said, grow in numbers rapidly. Some say, they grow in numbers 'rabbitly'. Let us see how the numbers grow. Suppose there are two rabbits, one male and one female. Each month they produce two baby rabbits. In two months baby rabbits are ready to breed. How many rabbits would be there at the end of the year?"

"Let us start with the initial pair in January.

January	1 pair	
February	2 pairs	(One of these pairs will start breeding only after two months, i.e. in April)
March	3 pairs	
April	5 pairs	
May	8 pairs	
June	13 pairs	
July	21 pairs	
. . .		
December	233 pairs	

"The number of rabbits each month follows the Fibonacci sequence," says Grandpa.

Magic 7 : Perfect Numbers

"That magic is perfect, isn't it?" Grandpa asks.

We nod our heads.

"So the time is ripe to introduce you to perfect numbers," Grandpa introduces a new topic. "Take the number 6. Its factors are 1, 2 and 3. Add the factors. The total is the number itself, i.e. 6. So 6 is a perfect number. It is the smallest perfect number.

A perfect number is one whose factors, including 1, add up to the number itself.

Some other perfect numbers are listed below.

$$28 = 1, 2, 4, 7, 14$$

Sum of the factors $= 1 + 2 + 4 + 7 + 14 = 28$

$$496 = 1, 2, 4, 8, 16, 31, 62, 124, 248$$

The sum of the factors is 496.

The next two perfect numbers in this sequence are 8128 and 33550336.

Note. All perfect numbers end in 6 or 8.

Magic 8: Link of Perfect Numbers with the Fibonacci Numbers

"There is a link between perfect numbers and the numbers of the Fibonacci series 1, 2, 3, 5, 8 ...," Grandpa perks up.

"Aha," we say no more.

"Take the number 3. It is a part of the Fibonacci series. Multiply 2 by itself (3 – 1) times, i.e. find 2×2.

$$2 \times 2 = 4 \qquad \ldots(a)$$

"Multiply 2 by itself thrice. We get 8. Subtract 1 from the result.

$$2 \times 2 \times 2 - 1 = 7 \qquad \ldots(b)$$

"Multiply (a) by (b)."

$$4 \times 7 = 28$$

"This is a perfect number. The factors are 1, 2, 4, 7 and 14. The factors add up to 28. This is the golden key to perfect numbers."

"Will it be so if we choose 5 in the Fibonacci series?" we ask.

"Watch," says Grandpa. "We pick 5. We have to find (5 – 1) th power, i.e. the fourth power of 2. It is 16. Now we take the 5th power of 2. It is 32. Subtract 1 from 32. We get 31. Multiply 31 by 16. We get 496".

"The factors of 496 are 1, 2, 4, 8, 16, 31, 62, 124, 248. The factors add up to 496," says Grandpa.

Magic 9 : Golden Ratio

"So you have the golden key to perfect numbers," Grandpa repeats the message, "but remember that perfect numbers have nothing to do with the Golden Ratio. The Golden Ratio is elusive. Very much like the golden deer which Lord Sriram chased. Yet there is a difference. The golden deer was an illusion. Not so with the Golden Ratio.

It is elusive because its precise value, like the pi, cannot be calculated. Yet it is very much there. It is about 1.6 (more accurately, it is 1.61803398) and is the ratio between two successive numbers in the Fibonacci sequence," says Grandpa.

He then writes down the ratio between successive numbers of the series.

$$\frac{2}{1} = 2.00; \ \frac{3}{2} = 1.50; \ \frac{5}{3} = 1.66; \ \frac{8}{5} = 1.60; \ \frac{13}{8} = 1.62; \ \frac{21}{13} = 1.61$$

"The Golden Ratio (GR) has a magic property," says Grandpa. "Here it is," Grandpa writes down the following steps.

$$GR-1 = 1.618-1 = .618 \text{ (Approx).}$$

$$\frac{1}{GR} = \frac{1}{1.618} = .618 \text{ (Approx).}$$

$$\text{Hence } GR-1 = \frac{1}{GR} \qquad \ldots(A)$$

Cross multiply (A).

$$GR \times GR - GR = 1$$

$$\text{Square of } GR = 1 + GR$$

"The GR has magic in it. Here is how the magic works. Draw a rectangle whose length and breadth are in the golden ratio. We could try it with a rectangle of sides 13 cm and 8 cm.

"Remove from this rectangle a square whose sides are 8 cm each. The remaining figure is a rectangle. Its sides are of length 8 cm and breadth 5 cm. 8/5 is also the Golden Ratio.

"Take away from this rectangle a square of sides 5. We get a rectangle of sides length 5 cm and width 3 cm. 5/3 is also approximately the Golden Ratio.

"Do this endlessly. Each time you take a square of sides whose length are equal to the breadth of the side of the rectangle, we are left with a new rectangle whose length and breadth keep the Golden Ratio. In other words, the Golden Ratio is for ever," Grandpa remembers something and shifts the focus. He asks, "Have you heard of Salvador Dali? He was a great painter. He used the Golden Ratio in some of his paintings."

"But nobody knows the exact value of the Golden Ratio. It is approximately 1.618, not exactly," Sunita points out.

"Well, the Golden Ratio is elusive like the eel. But we can represent it exactly. For that we must know something about surds," Grandpa tells us.

"Kurds, we know. They are a race of people who live in areas adjoining Iraq, Iran and Turkey. But Surds?" Ranjan shares with Grandpa the fact that the name is new to us. Totally new, if one may add.

SURDS

"Let me explain. The roots of only very few whole numbers are integers. The square roots of 4 is 2 and that of 9 is 3. But we cannot find the square roots of numbers between 4 and 9, i.e. 5, 6, 7 and 8, exactly. The cube root of 8 is 2 and that of 27 is 3. So the cube roots between 8 and 27 cannot be whole numbers. They lie between 2 and 3.

"Where then lies the square root of, say, 5. It lies between 4 and 9. So its square root must be more than 2 and less than 3. Its value is vague and is called a surd. It is represented as $\sqrt{5}$. The square of this surd is 5.

"The Golden Ratio is half of (1 + $\sqrt{5}$). It is the ratio that links numbers in the Fibonacci series. So one may say that the magic you perform with Fibonacci sequence lies in the Golden Ratio," Grandpa pauses.

"But it is the 7th number of the series that helps us when we try the magic. One look at the 7th number is enough for me to tell the total of the first 10 numbers of this series. Is that not be magic?" I conclude.

"Right," Grandpa is happy.

1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th		
1	1	2	3	5	8	13	21	34	55	$\xrightarrow{\text{total}}$ 143	= 13×11

4

CRYPTORITHMS

"Cryptography is the art of secret writing. The alphabets of the text are replaced by other alphabets, strings of numbers or alphabets according to well-defined rules. Those who love fun with numbers seek sums in which the digits are replaced by letters of the alphabets. Such sums are called CRYPTORITHMS. But cryptorithms are hard nuts to crack. They are for those who are not turned away by hard nuts," Grandpa strikes a note of caution.

"Are they all hard?" Ranjan sighs.

"No. We will begin with the easy ones. Study this simple question, where letters take the place of some of the numbers," says Grandpa.

Magic 1 : Secret Code 1

$$\begin{array}{r} 8\,A\,7\,B \\ +\,C\,5\,D\,8 \\ \hline E\,3\,4\,1\,2 \end{array}$$

"Start from the last column," says Grandpa and gives us the logic at each step.

B + 8 must give us a number ending in 2.

Since B is a single digit, the sum must be 12.

So we get B = 4. ($\because$ 12 – 8 = 4)

There is a carry over from the last column.

So 1 + 7 + D must give a number that ends in 1.

Therefore D = 3. [$\because$ 11 – 8 = 3]

Take the carry over of 1. We find that 1 + A + 5 must end in 4.

Take the carry over from the last column. So 8 + 1 + C must end in 3. So we get C = 4 and E = 1.

"This is also easy one," Grandpa writes down the problem.

Column *1 2 3*

$$\begin{array}{r} A\,B \\ B\,A \\ +\,B \\ \hline A\,A\,B \end{array}$$

"A = 1", I guess right away.

"Why?" asks Grandpa.

"The largest single digit numbers are 8 and 9. Their total is 17. A, the carry over from Column 2, is 1," I point out.

"Right," says Grandpa. Look at Column 3.

$$A + 2B = B$$

or $$A + B = 0$$

So $$B = 9$$

"That's quite easy," we think solving cryptorithms is just cakewalk.

Be a Magician

Find out the numbers that the letters represent in the sum

$$\begin{array}{r} 1\,A\;5\;8\;B \\ -\;9\;C\;D\;5 \\ \hline 6\;3\;3\;8 \\ \hline \end{array}$$

(Ans. A = 5; B = 3; C = 2; D = 4)

Magic 2 : Secret Code II

"Ah! Is that true? Let us try this problem."

$$\begin{array}{l|ccccccc|l} \textit{Column} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \\ & & & & N & T & B & L & \textit{Row 1} \\ & & & & \times & B & P & N & \textit{Row 2} \\ \hline & & & D & P & S & L & N & \textit{Row 3} \\ & & R & B & L & A & D & & \textit{Row 4} \\ & A & T & B & A & T & & & \textit{Row 5} \\ \hline & D & A & L & L & N & A & N & \textit{Row 6} \end{array}$$

We go through the collowing steps.

$D = A + 1$. (Column 1 : Why? Because D is sum of carry over from Column 2 which has to be 1.)

$L \times N = N$ (Column 7)

So N must be either 0 or 5.

But N cannot be 0. If that be the case, all letters in row 6 should all be 0s, i.e. the line should read N N N N N.

So $N = 5$

If N is 5, L must be an odd digit.

D is carryover of $N \times N = 5 \times 5 = 25$ + carry over from column 4.

So $D = 2$ or 3.

If $D = 2$,

$$A = 1 \text{ (Column 1)}$$

$$L = 9 \text{ (Column 6)}$$

$$1^* + P + L + A = L \text{ (Column 4)}$$

$$1 + P + 9 + 1 = 9 \text{ or } 19$$

$$11 + P = 19$$

$$P = 8$$

$$1^* + D + B + B = L \text{ (Column 5)}$$

$$1 + 2 + 2B = 9 \text{ or } 19$$

$$2B = 6 \text{ or } 16$$

B cannot be 8. (since $P = 8$)

So $B = 3$.

The problem now reads

---

* Carry over

$$
\begin{array}{rrrrrrrl}
 & & & 5 & T & 3 & 9 & \textit{Row 1} \\
 & & & \times & 3 & 8 & 5 & \textit{Row 2} \\
\hline
 & & 2 & 8 & S & 9 & 5 & \textit{Row 3} \\
 & R & 3 & 9 & 1 & 2 & & \textit{Row 4} \\
1 & T & 3 & 1 & T & & & \textit{Row 5} \\
\hline
2 & 1 & 9 & 9 & 5 & 1 & 5 &
\end{array}
$$

T is 7. (Why? Look at row 5. T is last digit of product of 9 and 3, i.e. 27)

S is 6 (Why? Take Column 5. There is a carry over of 1 from Column 6. Moreover T = 7.)

$$T + R = 11$$
$$7 + R = 11$$
$$R = 4$$

So we get the following number letter replacements.

A = 1; D = 2; B = 3; R = 4; N = 5; S = 6; T = 7; P = 8; and L = 9.

Be a Magician

$$
\begin{array}{rrrrrr}
 & & 5 & A & B & 7 \\
 & & & \times & C & 6 \\
\hline
 & 3 & 5 & 7 & 4 & D \\
 E & 7 & 6 & 5 & 6 & \\
\hline
F & G & D & H & K & 2
\end{array}
$$

The method is simple.

We note that 7×6 gives a number that ends in D. So D = 2.

There is a carry over of 4. So $6 \times B + 4$ gives a number ending in 4. So $6 \times B$ must give a number ending in 0. So we get B = 5.

Try to solve the sum on your own. You will find that A= 9; B = 5; C = 8; D = 2; E = 4; F = 5; G = 1; H =3; K =0.

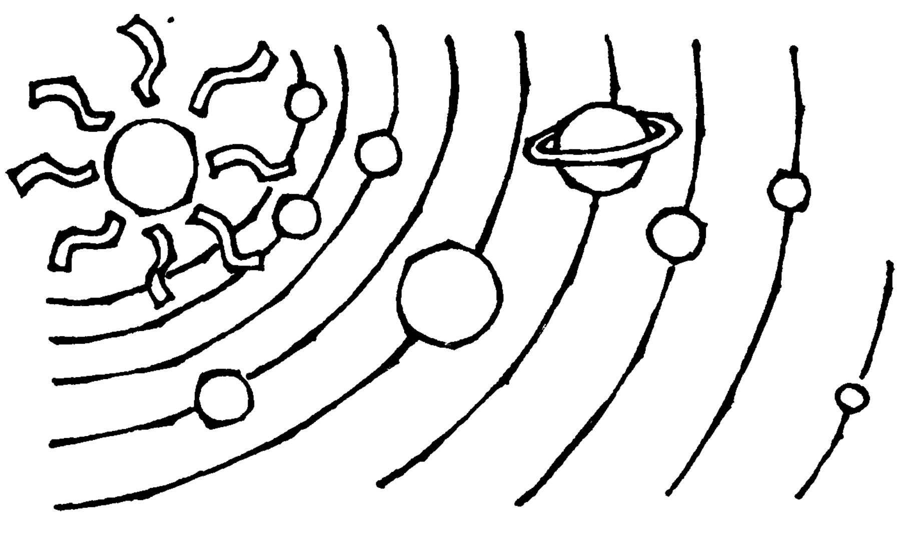
Magic 3 : Go with the Planets

"This problem is a hard nut to crack. Do you want to take it one?" asks Grandpa.

"We think we are up to any challenge. We are ready to scale the Everest, if we get a chance. How then could a silly problem, however hard it be, turn us away," I reply.

"The tough gets going when the going gets tough," Sunita drops a famous quote.

"Never say die is our policy," Ranjan adds.



"Good. Listen carefully. Here are five planets of the solar system whose letters have been replaced uniquely by the digits 0 to 9. One digit always stands for one letter and no other letter," says Grandpa.

Columns	*1*	*2*	*3*	*4*	*5*	*6*	*7*
		S	A	T	U	R	N
		U	R	A	N	U	S
			V	E	N	U	S
				M	A	R	S
	N	E	P	T	U	N	E

"Which is the planet closest to the sun?" Sunita asks.

"Is it Mercury?" I sound unsure.

"You will be sure if you remember this sentence," Grandpa waits till we are all ears and then spells out the sentence, "*Meghna's Very Elegant Ma Just Sat Urging New Plumbers.* Take the first one or two letters of each word and you get the order. Thus ME in MEGHNA stands for Mercury; VE leads us to Venus; E, of course, is short for our Earth," Grandpa chuckles.

"MA is for Mars; JU for Jupiter; SA for Saturn; UR for Uranus; NE for Neptune; and PL for Pluto," the three of us fill in the rest.

"You got it right. Well, let us now try magic and solve the sum," Grandpa gets ready.

"N = 1," I scream happily.

"Why?" Grandpa asks.

"Elementary," I reply, "There are two numbers in Column 2. The highest value possible for the numbers are 8 and 9. The carry over from Column 3 cannot be more than 2. So carry over of Column 2 that goes to Column 1 can be only 1. Hence N = 1."

"You are right," Grandpa rewrites the above sum thus.

Columns	*1*	*2*	*3*	*4*	*5*	*6*	*7*
		S	A	T	U	R	1
		U	R	A	1	U	S
			V	E	1	U	S
				M	A	R	S
	1	E	P	T	U	1	E

Grandpa explains every step thus.

Take Column 7

3S + 1 = E

Column S = 2? If so, we get E = 7. There is no carry over to Column 6.

Now look at Column 6.

2R + 2U is even. (Column 6)

Total of Column 6 is a number that ends in 1.

So 2R + 2U must be a number that ends in 0.

The 1 that comes as the total of Column 6 therefore comes as a carry over.

So S cannot be 2.

Let us now try S = 3.

We get from Column 7 that E = 0.

Go to Column 2.

S + U = 10 +E.

(For its carry over goes into Column 1)

E = 0

We then get S = 3; U = 6; R = 4 or 9.

Grandpa now rewrites the sum thus.

Case I. R = 4.

Column	1	2	3	4	5	6	7
		3	A	T	6	4	1
		6	4	A	1	6	3
			V	0	1	6	3
				M	A	4	3
	1	0	P	T	6	1	0

Carry over from Column 6 is 2.

Look at column 5. Carry over 2 + 6 + 1 + 1 + A = number ending in 6.

10 + A = number ending in 6.

So A has to be 6. But we have already taken U = 6.

Two letters can't stand for the same number.

So Case I fails.

Case II. R = 9

We have S = 3; U = 6; R = 9

Column	1	2	3	4	5	6	7
		3	A	T	6	9	1
		6	9	A	1	6	3
			V	0	1	6	3
				M	A	9	3
	1	0	P	T	6	1	0

Carry over to Column 4 is 1.

Total of Column 5 is a number ending in 6.

Carry over from Column 6 is 3.

So 3 + 6 + 1 + 1 + A = 6

11 + A must be 16.

So A = 5

Study Column 4.

1 (carry over from Column 5) + T + 5 + M = T

Subtract T from both sides.

We get 1 + 5 + M = number ending in 0.

So M = 4

Carry over from Column 4 is 1.

Replace A by 5.

1 (carry over from Column 4) + 5 + 9 + V = 10 + P

$$15 + V = 10 + P$$

$$V = P - 5$$

Let us now list the letters already identified.

E = 0; N = 1; S = 3; M = 4; A = 5; U = 6; R = 9.

So V = 2; and P = 7.

Only letter not identified is T. Only number left is 8.

So T = 8.

This gives us

E	N	V	S	M	A	U	P	T	R
0	1	2	3	4	5	6	7	8	9

Saturn =	358691
Venus =	20163
Mars =	4593
Uranus =	695163
Total =	1078610

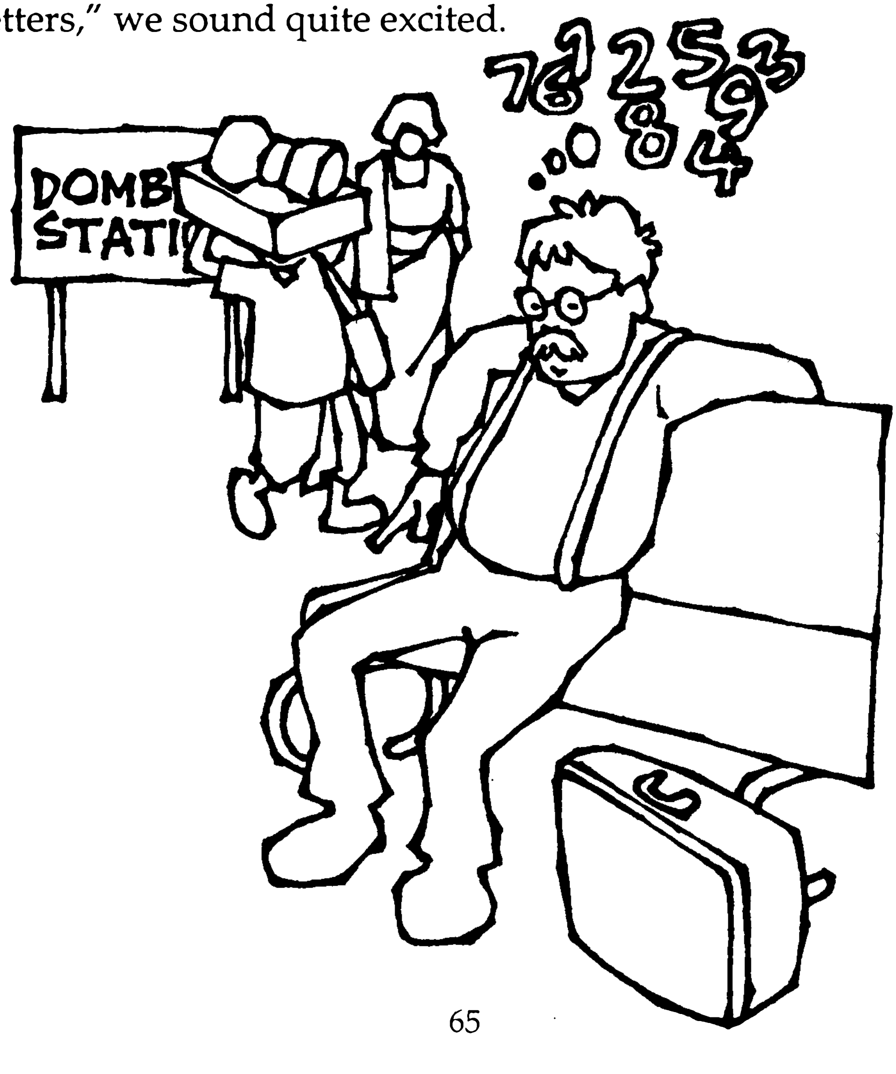
Replace 1078610 with letters. We get Neptune which is the only planet with a seven-letter name.

5

MIXED BAG

Magic 1 : Demlo Numbers

"There is magic in such sums. It is fun identifying the numbers behind the letters," we sound quite excited.



"Excitement is not your birthright alone. I remember Dattatreya Ramachandra Karpekar, a mathematical genius. He loved to play with numbers. One day, in 1923, he had a taste of excitement. He was waiting for a train at Dombivilli station in Mumbai. He sat on one of the benches there and closed his eyes. Numbers frisked around in his thoughts.

"Then a strange thought struck him. He thought of a number whose first and last parts, when added, give the digit or digits in between. Examples: 132; 264; 2553; 4773; 136653.

"Can we find such numbers?" Grandpa takes a pause.

"Can we?"

"Sometimes, yes. Take for example 234. Add the number to itself diagonally, i.e. write the number, with the last digit going under the last but one digit of the number above it, four times," Grandpa writes down the chart. It looks like this.

```
        2 3 4
      2 3 4
    2 3 4
+ 2 3 4
-----------
  2 5 9 9 7 4
```

Total comes to 2,59,974 (25 + 74 = 99)

"This is a Demlo Number," says Grandpa.

Be a Magician

Work out Demlo numbers starting with

(i) 57 (ii) 284

Magic 2 : Figure this Out

There are many numbers which can be expressed by using the numeral 4 four times and the arithmetical signs +, –, ×, ÷, $\sqrt{\ }$ (square root) and the decimal point.

For example,

$$11 = \frac{44}{\sqrt{4} + \sqrt{4}}$$

$$19 = \frac{4 + 4 - \cdot 4}{\cdot 4}$$

$$40 = \frac{\sqrt{4}}{\cdot 4}(4 + 4)$$

Be a Magician

Find such expressions for the numbers:

(i) 39 (ii) 50 (iii) 62 (iv) 84 (v) 100

Solution: There is no mathematical method for this. This is a test of one's ingenuity.

(i) $$39 = \frac{4 \times 4 - \cdot 4}{\cdot 4}$$

(ii) $$50 = \frac{4}{\cdot 4} \times \frac{\sqrt{4}}{\cdot 4}$$

(iii) $$84 = 44\sqrt{4} - 4$$

(iv) $$100 = \frac{44}{\cdot 44}$$

Magic 3 : Karpekar's Illegal Cancellation

"What's common between the fractions 16/64, 26/65, 19/95 and 1214/32171?" Grandpa writes down the fractions.

We are stumped. We examine the fractions, but get no closer to the answer.

"Drop numbers (not digits) which are common to the numerator and denominator of each fraction," says Grandpa.

We do that. The fractions now read:

$$\frac{1}{4} \qquad \frac{2}{5} \qquad \frac{1}{5} \qquad \frac{14}{371}$$

(In the last fraction 21 is struck off.)

"Look at $\frac{16}{64}$ and $\frac{1}{4}$. Are these fractions equal in value?" he asks.

We try the cross multiplication method. Then we shout, "The fractions are equal."

Be a Magician

Test the remaining three set of fractions, using the cross multiplication method. What do you find? The value of each set of fractions is the same.

This too was first reported by Karpekar.

Magic 4 : Binary Numbers

"Karpekar was a genius," says Grandpa.

"So are computers," I break in.

"Computers are dunces. They can't handle numbers, as we do," says Grandpa.

"Are you joking Grandpa? Don't you know how quick the computers are in handling numbers. They add, subtract, multiply and divide very large numbers in fractions of a second," we say.

"None disputes that fact. Yet, it is also true that the computer cannot identify numbers other than 0 and 1. These are the two numbers in the binary system." says Grandpa.

"What is binary system?" we ask.

"It is very much like the normal number system that we use. In the system we go by multiples of 10."

"Remember that any number, raised to the power 0, is 1"

9 = 9 × 10 raised to the power 0 (9×10^0)

17 = 1 × 10 raised to the power 1 (1×10^1) +
7 × 10 raised to the power 0 (7×10^0)

348 = 3 × 10 raised to the power 2 (3×10^2) +
4 × 10 raised to the power 1 (4×10^1) +
8 × 10 raised to the power 0 (8×10^0)

"While using the numbers, we take the powers of 10 for granted. And we refer to 3 as occurring at the hundreds position, the 4 at the tens position and 8 at the ones position. Suppose we replace 10 by 2. What happens?" Grandpa starts recording the steps on paper.

10 = 8 + 2
= $1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$
= 1 × 2 raised to the power 3 +
0 × 2 raised to the power 2 +
1 × 2 raised to the power 1 +
0 × 2 raised to the power 0.

So in the binary system, powers of 2 are omitted. The normal number 10, becomes 1010.

"What will be the number 25 in binary system?" asks Grandpa

$$
\begin{aligned}
25 &= 16 + 8 + 1 \\
&= 1 \times 2 \text{ raised to the power } 4 + \\
&\quad 1 \times 2 \text{ raised to the power } 3 + \\
&\quad 0 \times 2 \text{ raised to the power } 2 + \\
&\quad 0 \times 2 \text{ raised to the power } 1 + \\
&\quad 1 \times 2 \text{ raised to the power } 0.
\end{aligned}
$$

"So in the binary system, 25 is read as 11001."

"Is there an easy method to convert normal numbers into the equivalent binary numbers?," we ask.

"There is. Here we go. Let us try to find the binary number that stands for 25," Grandpa starts the demonstration.

Step I

```
2 ) 25 ( 12
    24
    --
     1
```

Step II

```
2) 12 ( 6
   12
   --
    0
```

Step III

```
2 ) 6 ( 3
    6
    -
    0
```

Step IV

```
2 ) 3 ( 1
    2
    -
    1
```

"We stop at this point. We start with the last quotient, i.e. 1. Then we write down the remainders at each stage. The binary number which stands for the normal number 25, obtained by this method, is 11001," says Grandpa.

Be a Magician

Convert the following normal numbers into binary numbers.

(i) 73   (ii) 139

Ans. (i) 1001001

(ii) 10001011

Magic 5 : Digital Roots

"So we need to know only how to write 0 and 1, when we use binary numbers," we say.

"That is where its magic lies. Where does the magic lie in digital roots?" Grandpa stops on seeing the blank looks in our eyes.

"Define the term, digital root," Sunita asks.

"The digital root of a perfect cube is the remainder we get when we divide it by 9. The remainder is always 0 or 1 or 8," says Grandpa.

"Why?" we ask.

He explains. It runs like this.

"Every number is a multiple of 3; or a multiple of 3 + 1; or a multiple of 3 + 2. Why?"

"Because a multiple of 3 + 3 is itself a multiple of 3," we shout in chorus.

"So every number can be expressed as 3a, 3a + 1 or 3a + 2," Grandpa writes down the following chart while we watch.

Case 1. Number is of form 3a.

$$3a \times 3a \times 3a = 27 \times a^3$$

When this number is divided by 9, the remainder is 0.

Case 2. Number is of form 3a +1.

$(3a + 1) \times (3a + 1) \times (3a + 1) = 27 \times a^3 + 27 \times a^2 + 9a + 1.$
Divide this by 9. Remainder is 1.

Case 3. Number is of form 3a + 2.

$(3a + 2) \times (3a + 2) \times (3a + 2) = 27 \times a^3 + 54 \times a^2 + 36a + 8$

Divide this by 9. Remainder is 8.

Grandpa grins happily. Suddenly he shifts tack and asks, "I know you love to play with marbles. Can we have some fun with marbles?"

"Why not?" we readily agree.

Magic 6 : Fun With Pyramids

"Pick up a pile of marbles. Find a square-shaped lid. Turn it up so that the rim of the lid forms a border. Fill it up tightly with marbles. You now have a base of marbles. Count the number of marbles along the rim. (Say 8). You will find the same number of marbles, on all four sides, because you have a square base.

"Now build the pyramid, by forming layers of marbles. Call a friend. Point out to him the fact that the pyramid has a square base. Also tell him that 8 marbles go along each side of the base. Ask him to guess the number of marbles which form the pyramid, without dismantling the pyramid. He will say that nobody could guess that. Can you?" asks Grandpa.
We shake our heads. We can't work out the answer. It is beyond us.

"The answer is easy to work out," says Grandpa.

He writes down the following steps.

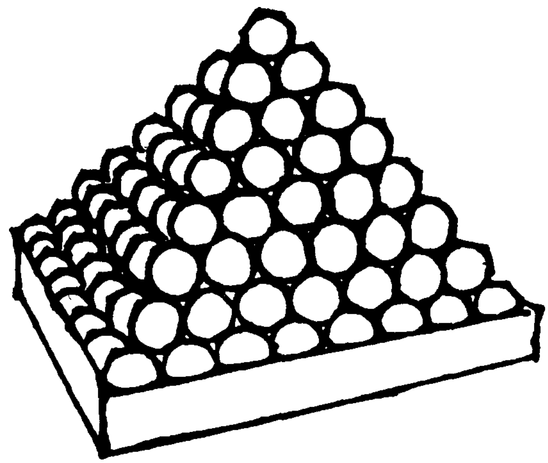
The total number of marbles, at the base $8 \times 8 = 64$.

"The next layer has 7 balls all along the sides," says he.

The number of marbles in this layer is $7 \times 7 = 49$.

"What? about the next layers" he asks.

"It must be 6×6 marbles," we guess.



"You got it right," says Grandpa.

Now we see light.

The total number of marbles held by the lid is $64 + 49 + 36 + 25 + 16 + 9 + 4 + 1$. It is 204.

"One strange fact pops up, at this stage," says Grandpa.

We can see nothing.

"Look at 204," says Grandpa. "I shall play with it," he goes back to paper and pencil.

$$204 \times 6 = 1224$$

But $$1224 = 8 \times 9 \times 17.$$

So $$204 = 8 \times 9 \times 17 \text{ divided by } 6$$

But 8 is number of marbles along one side of the base.

9 is $(8 + 1)$; and 17 is $(2 \times 8) + 1$.

"So, we can make a quick calculation by following these steps."

1. Note the number of marbles (in this case it is 8) along one side of the base.

2. Multiply the number obtained at step 1 by the next number in ascending order in the number series. (In this case, it is 8 × 9 = 72)
3. Take the number obtained at step 1. Double it. Add 1. (In this case, the starting number is 8. Its double is 16. Add 1. We get 17)
4. Multiply the number obtained at step 2 by the number obtained at step 3. (In this case, it is 72 × 17 = 1,224).
5. Divide the result at step 4 by 6. (In this case, we get 204)

Be a Magician

Verify by working out the number of marbles in a pyramid with a square base where the number of marbles along the rim of the bottom layer is 10. Try both methods.

First Method

Find square of all numbers from 1 to 10 and find the total. What is the total? (It will be 385)

Second Method

Try out the formula. Multiply 10 by 11. We get 110. Multiply this by 21. The result is 2310. Divide this by 6. We get 385. The formula works.

Sum of Squares of 'n' Natural Numbers

"So," Grandpa finds his voice again, "if 'n' marbles go on each side of the square, the pyramid, we can say, with certainty, holds (1 × 1) + (2 × 2) + (3 × 3) + ... + (n × n) marbles. The formula says it will be n × (n + 1) × (2n + 1) divided by 6. That gives us yet another interesting lead. The sum of the squares of the first 6 digits, 1 to 6, is

$$\frac{6 \times 7 \times 13}{6} \text{ i.e. } 91.$$

Let us check it. 1 + 4 + 9 + 16 + 25 + 36 = 91

Lo Presto! The answers are identical.

Be a Magician

(i) Find the sum of the squares of the first 15 natural numbers.

(ii) Find the number of marbles in a pyramid with a square base which holds at the bottom layer along one edge (a) 9 marbles (b) 12 marbles (c) 20 marbles.

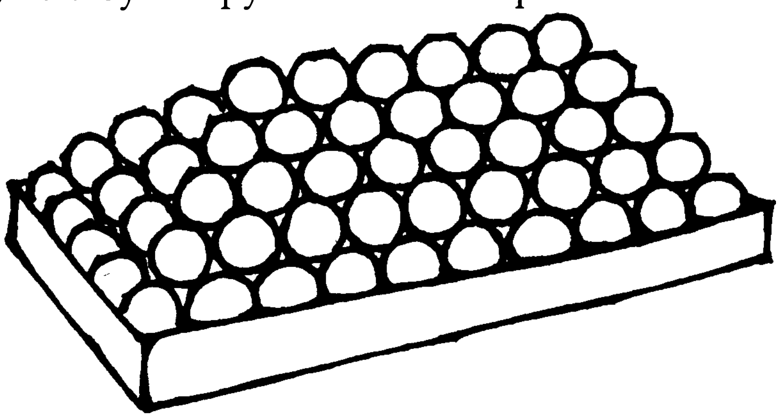
(iii) Find the sum of squares of natural numbers from 7 to 24.

(Remember that sum of the squares from 7 to 24 is the difference between the sum of the squares of numbers 1 to 24 and of numbers 1 to 6).

Try by actual addition. Test with formula.

When Base is Rectangular

"Suppose in the previous problem, the base is not a square, but a rectangle. Along the length of the bottom layer goes, say, 10 marbles. Along the width goes 5 marbles. How will we estimate the number of marble, held by this pyramid?" Grandpa asks.



"The number of marbles, at the base, is $10 \times 5 = 50$," we say.

"Right. How about the layer above the first?" he asks.

"The next layer holds $9 \times 4 = 36$ marbles," we vie with each other to be the first to give the answer.

"The next layer holds ... "

"$8 \times 3 = 24$ marbles," we say.

We now know how to make the right estimate. The total number of marbles, held by the pyramid with the rectangular base, of 10 marbles along the longer side and 5 along the shorter side is

$10 \times 5 + 9 \times 4 + 8 \times 3 + 7 \times 2 + 6 \times 1$

i.e. $50 + 36 + 24 + 14 + 6 = 130$ marbles.

Do we find a pattern here, too?

What is 130? It is 780/6.

$780 = 5 \times 6 \times 26$.(5 is the number of marbles along the width)

$$26 = (30 - 5 + 1) = (3 \times 10 - 5 + 1)$$

(10 is the number of marbles along the length)

"Can we guess the number of marbles, held by a rectangular pyramid, with 'n' marbles along the longer side of the base and 'm' along the shorter side of the base? Can we say that the number of marbles will be $m \times (m + 1) \times (3n - m+1)$ divided by 6?" asks Grandpa.

"Can we test the formula?" we ask.

"Why not?" Grandpa sets down a couple of questions. We try them out. The formula works. Want to have some practice! Here are the questions.

Be a Magician

Find the number of marbles in a pyramid formed by marbles resting on a lid with

(i) 8 marbles along the length and 6 along the width at the base.

(ii) 16 marbles along the length and 12 along the width at the base.

(iii) In (i) find marbles held between layers 3 and the top.

(iv) In (ii) find marbles held between the base and the 4th layer.

Note:

This leads us to the formula

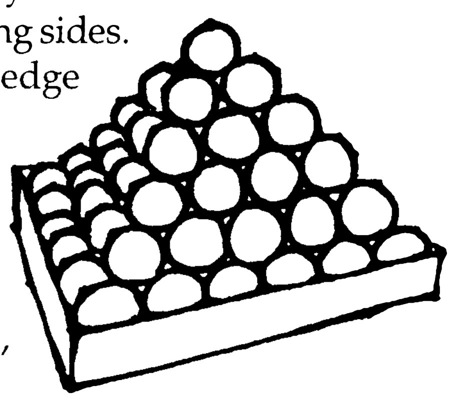
$$n \times m + (n-1) \times (m-1) + (n-2) \times (m-2) + \ \ldots \ + (n-m+1) \times 1 = m \times (m+1) \times (3n-m+1)/6$$

When the Base is an Equilateral Triangle

"Suppose the base is shaped like a triangle whose sides are all equal. How many marbles are held by the pyramid if 6 marbles go along one edge of the triangle?" Grandpa asks.

We try to calculate, but fail.

"Along one edge we get 6 marbles," says Grandpa. "Along the next edge we get only 5 marbles. Why? Because one marble at the vertex where the two edges meet is shared by the two. And along the third edge we get 4 marbles. You know why. The side shares one marble of each of the adjoining sides. So total number of marbles along the edge is 6 + 5 + 4.



"The inner line of the base holds, by the same logic 3, 2 and 1. So total number at the base is 6 + 5 + 4 + 3 + 2 + 1, i.e. 21.

"The layer above holds 5 + 4 + 3 + 2 + 1, i.e. 15

"The layer still above holds 4 + 3 + 2 + 1, i.e. 10

"The layers further up hold 3 + 2 + 1, i.e. 6 marbles; and 2 + 1, i.e. 3 marbles; and finally 1 marble.

The total comes to 21 + 15 + 10 + 6 + 3 + 1, i.e. 56 marbles."

"Don't say this too can lead to a formula?" Ranjan jokes.

"It can. In fact it does," says Grandpa. Let us see the different steps.

"What is 21? It is the sum of numbers 1 to 6. It is also 6 × 7 divided by 2. What is 15? It is 5 × 6 divided by 2 . . . " Grandpa peers at us.

Then we get a little insight. 4 × 5 divided by 2 is 10; 3 × 4 divided by 2 is 6; 2 × 3 divided by 2 is 3; and 1 × 2 divided by 2 is 1. These are the numbers of marbles held at the layers as we move up to the top of the pyramid.

So we get a formula:

$$1 + 2 + 3 + 4 + \ldots + (n - 1) + n = n \times \frac{(n + 1)}{2}$$

Then we go to the next step of the sum.

The total number of marbles is

$$\frac{1}{2}(6 \times 7 + 5 \times 6 + 4 \times 5 + 3 \times 4 + 2 \times 3 + 1 \times 2) = 56$$

What is 56? It is 6 × 7 × 8 divided by 6.

Does it give us a hint? Does it mean that the total number of marbles held by a pyramid, whose base is an equilateral triangle and each edge holds 6 marbles, is 6 × 7 × 8 divided by 6?

"If n marbles go along one edge of an equilateral triangle, will the pyramid of marbles that it holds be $\frac{n \times (n + 1) \times n + 2}{6}$?" we ask.

"You got it right," says Grandpa.

Be a Magician

Test this formula with pyramids with base of equilateral triangles which can hold

(i) 10 marbles (ii) 15 marbles

What are the results computed by actual calculation? Do they agree with the result obtained by using the formula?

The results are (i) 220 marbles; (ii) 680 marbles

Note:

This gives us a formula.

$$n \times (n + 1) + (n-1) \times n + (n - 1) \times (n - 2) +$$

$$\ldots + 2 \times 3 + 1 \times 2 = n \times (n + 1) \times \frac{(n + 2)}{6}$$

Magic 7 : Sum of Natural Numbers

"Is there not an easy method to sum up the first n natural numbers?" we ask.

"There is. Have you heard of Carl Frederick Gauss? His teacher asked the class — he was about 9 at that time — to find the sum of all natural numbers from 1 to 100. The children started adding. Gauss stood up within a minute to with the answer 5,050. The teacher was taken aback. He asked: 'How did you find the answer?'

"Sir," said Gauss. "I grouped the numbers in pairs.

(1 + 100), (2 + 99), (3 + 98) ... (48 + 53), (49 + 52) and (50 + 51). I got 50 pairs each adding up to 101. 101 × 50 gave me the total as 5,050.' The teacher was stumped," says Grandpa.

"Is there no formula?" Ranjan asks.

"Is it not evident?" Grandpa returns to paper and pencil. Here is what he writes.

$$5{,}050 = 50 \times 101 = \left(\frac{100}{2}\right) \times 101 = \left(\frac{(100 \times 101)}{2}\right)$$

"The sum of the numbers from 1 to 100 is 100 × 101 divided by 2. Now tell me, what is the sum of numbers from 1 to 30?" Grandpa asks.

We take very little time to find the answer. We say, "It is 30 × 31 divided by 2, i.e. 465."

"Good. So the sum of the first n natural numbers is . . . "Grandpa" writes down the total. It is $\frac{n(n+1)}{2}$.

Be a Magician

Find the total of numbers from

(i) 1 to 80 (ii) 7 to 35 (iii) 12 to 54

(Use formula. Check by actual summation.)

(Hint: In (ii) find sum of numbers from 1 to 6 and 1 to 35. Subtract 1st from 2nd.)

Magic 8 : Indeterminate Equations I

"Divide the number 316 into two parts so that one part is divisible by 11 and the other by 13", Grandpa asks.

"The numbers are 121 and 195", he adds.

"How is this obtained?" we asked

He goes on.

He prepared a list of multiples of 11 and multiples of 13.

Multiples of 11	*Multiples of 13*	
(a)	(b)	
11	13	
22	26	
33	39	
44	52	
55	65	
66	78	
77	91	
88	104	
99	117	
110	130	
121	143	
132	156	
143	169	
154	182	
176	195	
187	208	
198	221	and so on

Take up each number from column (a). Thus we start with 11.

Add this to every number in (b). Do we get 316?

If not take the next number. Add it to every number in (b).

At some stage, we will get the total as 316. We get 121 from list (a) and 195 from list (b).

$$121 + 195 = 316.$$

121 is divisible by 11. 195 is divisible by 13.

An Easier, Alternate Method to this Problem

"Take the product of the two factors, 11 and 13," Grandpa narrates.

We get 143.

"Take off 143 from 316, as many times as possible.

$$316 - 143 = 173$$
$$173 - 143 = 30 \quad \ldots \text{(a)}$$

"Take the difference between the two factors.

$$13 - 11 = 2 \quad \ldots \text{(b)}$$

"Divide (a) by (b). We get 15... (c)

"Multiply the bigger of the two factors, (in this case 13) by (c). We get 195, which is one part.

"Subtract 195 from 316 to get the other part. It is 121."

Be a Magician

Try the following sums, using both the long and the short cut methods.

(i) Divide 450 into two parts, one divisible by 13 and the other by 17.

(ii) Divide 347 into two parts, one divisible by 7 and the other by 9.

Note: Does the short cut method work always? If not, what are the limitations?

Can you divide a large number into two parts, each part divisible by two different numbers, one of which is a factor of the other?

Magic 9 : Indeterminate Equations II

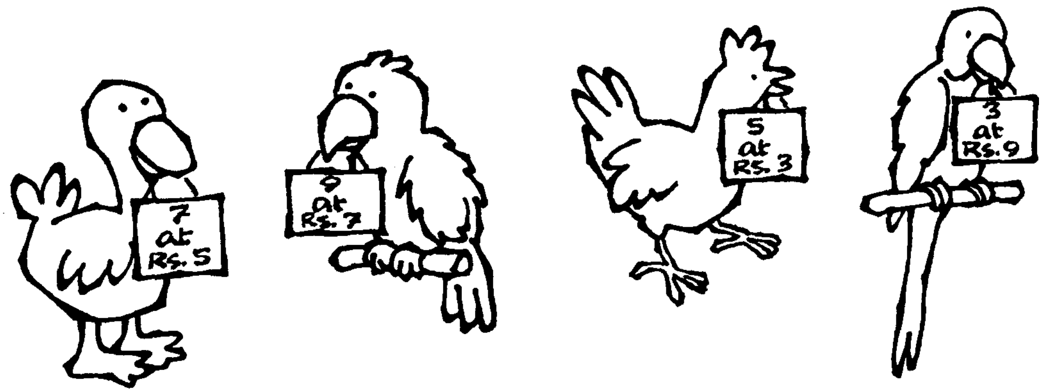
"Here is a problem that Bhaskaracharya of yore set down to his students," says Grandpa. "It is a rather difficult problem and only if you use the magic that goes with logic then you can solve it."

"Give us the problem," we are keen.

"Suppose I give you Rs 100. I want you to get me 100 brids with the amount. Remember that 5 hens cost Rs 3; 7 ducks cost Rs 5; 9 parakeets cost Rs 7; and 3 parrots cost Rs 9", Grandpa races on while we note down the details.

We sit dazed. We don't know where to start.

Grandpa then helps us.



"Here is a chart that may help you," he prepares the chart as under.

	A	*B*	*C*	*D*	*E*	*F*	*G*	*H*
	No. of hens	*Cost*	*No. of Ducks*	*Cost*	*No. of Parakeets*	*Cost*	*No. of Parrots*	*Cost*
P	5	3	7	5	9	7	3	9
Q	10	6	14	10	18	14	6	18
R	15	9	21	15	27	21	9	27
S	20	12	28	20	36	28	12	36
T	25	15	35	25	45	35	15	45
U	30	18	42	30	54	42	18	54

"That should do," says Grandpa.

"Why?" we ask.

"Because we have only Rs 100 and we need only 100 birds. So we can't buy too many parrots. They are costly. 18 parrots take away Rs 54," Grandpa uses logic.

Then he points out another logic. "Look at Column A and Column D. The figures are multiples of 5," he pauses before adding, "Now begins the real hard work. We have to choose pairs of values

from (A, B), (C, D), (E, F) and (G, H) in such a way that A + C + E + G = B + D + F + H = 100," Grandpa purses his lips.

We struggle till we land on the following combination.

R 15 hens	Rs 9
S 28 ducks	Rs 20
T 45 parakeets	Rs 35
S 12 parrots	Rs 36
Total = 100 birds	Rs 100

We can't believe our eyes. We got it right.

"Thank you," we tell Grandpa.

"Thank Bhaskaracharya," he jokes.

Magic 10 : Find N

"A large number, (Let us call it N) has 7 as the last digit. If this 7 is shifted to the extreme left, as the first digit, we get double the number N. Find N," Grandpa says.

$$N = 368421052631578947.$$

"How?" we asked. Grandpa explains.

$$N = \ldots\ldots\ldots\ldots\ldots 7$$

$$2N = 7\ldots\ldots\ldots\ldots\ldots$$

Since N ends in 7, 2 N must end in 4. ($7 \times 2 = 14$)

$$\therefore\ 2N = 7\ldots\ldots\ldots\ldots\ 4$$

So $\quad N = \ldots\ldots\ldots\ldots\ 47$

Now $\quad 2N = 7\ldots\ldots\ldots\ldots\ 94$

$$N = \ldots\ldots\ldots\ldots\ 947$$

Applying this logic, we alternate between 2N and N till we get the number.

We finally get $\quad N = 368421052631578947$

$$2N = 736842105263157894$$

Teasers

Here are a few teasers that Grandpa shared with us.

1. A professor of mathematics was hit by a taxi and seriously hurt. Before he sank into coma, he gave the following details about the taxi's number. It is a 4-digit number which when made to stand on its head, would still read the same. It is a perfect square. Its square root is a prime. When the square root is added to another prime, you get a perfect square. Find the number.

2. $1 \times 1 + 2 \times 2 + 2 \times 2 = 3 \times 3$

   $2 \times 2 + 3 \times 3 + 6 \times 6 = 7 \times 7$

   $3 \times 3 + 4 \times 4 + 12 \times 12 = 13 \times 13$

   $4 \times 4 + 5 \times 5 + 20 \times 20 = 21 \times 21$

   Does this give a rule? Start with two consecutive numbers a and a + 1. The product of the two numbers is $a \times (a + 1)$

   Take the square of a, (a +1) and $\{a \times (a + 1)\}$.

   Add. It will be the square of $\{a \times (a + 1) + 1)\}$

   Test it with 7 and 8. Their product is 56.

Square of 7	=	49
Square of 8	=	64
Square of 56	=	<u>31,36</u>
Total	=	<u>32,49</u>

   Square of 57 = 32,49

3. Can you get 1996 using the digits in the year, i.e. 1, 9, 9 and 6 and mathematical symbols?

(Ans. $1996 = [1 + (9 \times 9) + (9 \times 9 \times 9) + (6 \times 6 \times 6 \times 6) + \sqrt{1} + \sqrt{9}] - 19 - 96$)

4. Can you find out a number with the following qualities?

   (a) Square it, add 5 to the square, it still is a perfect square.

   (b) Square it and subtract 5 from this number. The resulting number is also a perfect square.

Ans. The number is $\frac{41}{12}$.

Square it. We get 1681/144. Add 5.

$$\frac{1681}{144} + \frac{5}{1} = \frac{2401}{144}.$$

It is the square of $\frac{49}{12}$.

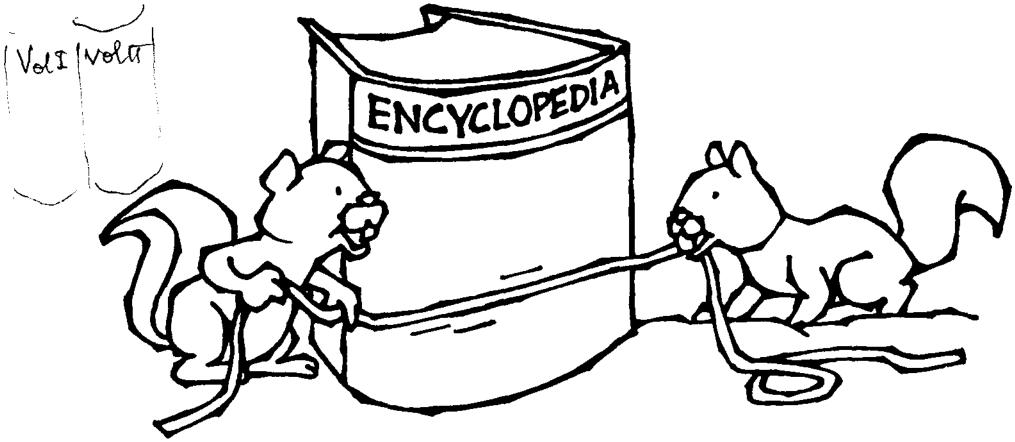
Square the number $\frac{41}{12}$. We get $\frac{1681}{144}$. Subtract 5 from it.

$$\frac{1682}{144} - \frac{5}{1} = \frac{961}{144}.$$ It is the square of $\frac{31}{12}$.

$\frac{41}{12}$ is unique. Perhaps it is the only number which shows this feature.

5. An encyclopedia with one volume for each letter of the English alphabet is shelved in the usual manner with the letters A to Z from left to right. The two covers, (front and back) of each volume measures 6 mm. The pages in between are 12 mm thick.

Find the distance between the first page of Volume A and the last page of Volume B.



Ans. Most people would say 36 mm.The answer is actually the distance between the front cover of volume A and the back cover of volume B, i.e. 12 mm. Try it if you have doubts, actually at any library. (I read this in Yuva Bharati of Sept 84 sent by Mr R. Kesavan)

6. There are two numbers A and B. The sum of the digits of A is same as the sum of the digits of B. It is also the sum of the digits of the number we obtain by subtracting B from A. Find A and B.

Ans. The first number (A) is 9 8 7 6 5 4 3 2 1

The second number (B) is 1 2 3 4 5 6 7 8 9

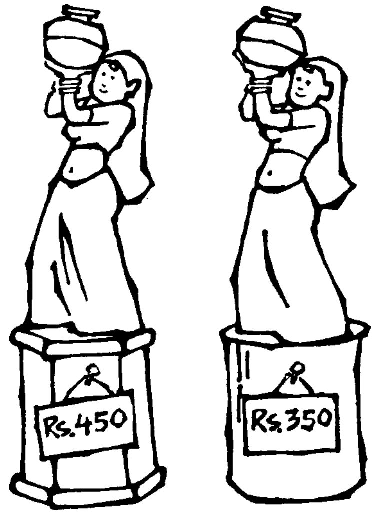
The difference is 8 6 4 1 9 7 5 3 2

Test the sum of the digits. We always get 45. A is the numbers 9 to 1 in descending order. B is the numbers 1 to 9 in ascending order. In the difference, all the figures 1 to 9 appear once each, though the order is different.

7. 64 coins are held in four uneven heaps. I take as many coins from the first heap as there are in the second and add them to the second. Now I take as many coins from the second as there are in the third heap and add them to the third. I take as many

coins from the third heap as are held by the fourth heap and add to the fourth. Finally, from fourth, I take as many coins as are there in the first heap and add to the first one. Now all the heaps have 16 coins each. How many coins did each heap hold at the start?

(**Ans.** Heap 1 holds 23; heap 2 holds 15, heap 3 holds 14; and heap 4 holds 12)



√8. A statue mounted on a steel block costs Rs 450. The statue mounted on block of wood costs Rs 350. If cost of wood is half that of steel, what is the price of the statue?

(**Ans.** Rs 250).

9. Find the two smallest numbers, the difference of whose cubes is a square and the difference of the squares is a cube.

(**Ans.** 6 and 10)

10. Rajender is 'A' years old. The square of his age and the cube of his age contains all the digits just once each. Find Rajinder's age?

Hint:

(i) The square of his age must have four digits; and the cube six digits. For between them, they have to hold all the digits. So the square of his age must be more than 1,000. Remember that 1,024 is the first perfect square after 1,000. Hence Rajinder must be at least 32 years old.

(ii) Since the digits of the squares and the cubes must be distinct, they cannot share a common digit at the unit positions. So his age cannot end in 0, 1, 5, and 6. The two hints and some hard work lead us to the right answer.

(**Ans.** 69 × 69 = 4,761. 69 × 69 × 69 = 3,28,509

Note that all the ten digits occur just once.

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